Sources of Liquidity and Liquidity Shortages

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ABSTRACT

We develop a model of liquidity shortages that incorporates a general equilibrium feature of liquidity: when banks hold more liquidity, other agents in the economy hold less of it and will supply less in times of crisis. We show that the private holdings of liquidity at banks are inefficient, with the direction of the bias being determined by the characteristics of the suppliers of liquidity to banks. Minimum liquidity requirements for banks may reduce welfare; in such cases interest rate policies that stimulate the ex-post supply of liquidity can restore efficiency. Overall, our results show that optimal liquidity policies critically depend on a financial institution’s (marginal) source of liquidity and will hence differ across institutions of different types.

1. Introduction

The experience of the Global Financial Crisis has shown that liquidity is central to the well-functioning of the financial system. However, due to its public good character, agents in the financial system may not have sufficient incentives to invest in it. This view is reflected in the Liquidity Coverage Ratio (LCR) of Basel III, which stipulates minimum liquidity levels for banks to cover potential liquidity needs. In this paper we show that the desirability of such regulation depends crucially on the suppliers of liquidity to banks in times of crisis. The reason is that higher liquidity holdings at banks are likely to lower liquidity in other parts of the economy, and reduce the supply of liquidity to banks in times of need, with unclear implications for welfare. Any frictions to raising liquidity ex-post (when it is needed) hence have to be weighted against frictions from raising and maintaining liquidity ex-ante. Forcing higher liquidity holdings at banks with liquidity suppliers that have a good ability to overcome frictions in times of crises may then lower the ability of the financial system to deal with liquidity problems.

We develop a model in which banks interact with an “investor” sector in order to deal with uncertain future liquidity needs (these liquidity needs can be interpreted as arising from the projects of firms funded by banks). Banks can raise liquidity in anticipation of liquidity needs or once a need materializes. There are frictions associated with both ex-ante and ex-post liquidity, arising from agency and informational problems, respectively. These frictions make both channels of obtaining liquidity costly. Welfare losses arise in the event of banking sector-wide liquidity shortages. Such shortages can materialize for two reasons in our model: an insufficient borrowing capacity of the banking sector or an insufficient supply of liquidity by investors.

We show that the private incentives of banks and investors leads to inefficient precautionary liquidity holdings at banks. The source of the inefficiency is a fire-sale externality: when it liquidates projects, an individual bank does not internalize the increased costs to other liquidating banks. Importantly, we find that precautionary liquidity can either be insufficient or excessive. To understand this result, suppose first that there are costs to raising liquidity ex-post – but no cost ex-ante. In this case, storing liquidity within the banking sector – as opposed to the investor sector – maximizes the amount of liquidity that can be used at banks to cover liquidity shocks. Precautionary liquidity holdings at an individual bank then increase the ability of the banking sector to deal with liquidity shortages, resulting in fewer liquidations in the economy. As the bank does not internalize the external effect of this, its holdings of liquidity will be insufficient. The exact opposite result is obtained when the cost is to raising liquidity ex-ante, for example because managers will divert free liquidity. In this case, precautionary liquidity reduces the overall amount of liquidity that can be generated by banks as it lowers the supply of investor liquidity ex-post by a larger amount. Private holdings of liquidity in the banking sector will then exceed the socially optimal amount. When there are both ex-ante and ex-post costs, we show that the bias in private liquidity holdings (insufficient or excessive) depends on the relative size of the two costs.

Liquidity regulation in the form of minimum liquidity requirements is hence not necessarily desirable. For example, financial institutions that tend to raise funds from a pool of relatively sophisticated investors face relatively low costs of accessing this pool in crisis times (as sophisticated investors can more easily overcome informational frictions). A unit of liquidity in the pool is thus particularly valuable. Since precautionary liquidity reduces the pool of liquidity available in a crisis, minimum liquidity requirements may lower welfare as a result. We show that in such cases efficiency can be improved through interest rate policies that encourage the private supply of liquidity ex-post, for

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https://doi.org/10.1016/j.jfi.2020.100869
Received 19 October 2018; Received in revised form 3 March 2020; Accepted 27 March 2020
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Please cite this article as: Charles M. Kahn and Wolf Wagner, Journal of Financial Intermediation, https://doi.org/10.1016/j.jfi.2020.100869
example, by providing public liquidity at low rates during times of stress in order to reduce the attractiveness of relying on ex-ante liquidity.

Our analysis suggests that the desirability of standard liquidity policies depends crucially on the (marginal) sources of liquidity. Optimal liquidity policies will hence differ across financial institutions of different types. This has profound implications for financial regulation which has tended to regulate liquidity based on the institution where it is held (i.e., liquidity requirements for banks) but not according to the institution’s liquidity source.

Banks’ incentives to invest in liquidity are of long-standing interest to the banking literature. A key theme is that banks underinvest in liquidity – due to the public good character of liquidity (e.g., Bhattacharya et al., 2009). Our model shares key features with the literature (such as the public good character of liquidity); the reason we obtain different results is due to two ingredients. First, there is a common pool of liquidity across periods, hence raising more ex-ante liquidity lowers the ability to raise liquidity ex-post. Second, we allow for a friction to ex-ante liquidity such that not all funds that are raised from investors are available to satisfy liquidity needs should a liquidity shock hit later.

Our interest in the allocation of liquidity relates to work by Holmström and Tirole (1998) and Bolton et al. (2011). Holmström and Tirole study an economy where consumers and firms can only imperfectly pledge incomes. They show that there is scope for public provision of aggregate liquidity as the financial system cannot insure itself against aggregate shocks (a similar result is obtained by Gorton and Huang, 2004). Our analysis – which also considers limits to pledgability – by contrast focuses on inefficiencies in the private allocation of liquidity between the financial and the private sector. Bolton et al. consider a setting where informational asymmetries about asset quality vary over time and use this to study the optimal timing of asset sales by banks. In their paper banks face higher adverse selection costs when they experience liquidity needs (this corresponds to large ex-post liquidity costs in our setting). When adverse selection is sufficiently large, they show that there can be an (inefficient) equilibrium in which banks sell assets early and hold more liquidity than in the efficient equilibrium. Whereas the inefficiency in Bolton et al. arises due to adverse selection, the source of the inefficiency in our model is a fire sale externality (e.g., Stein, 2009).

Our analysis also provides insights into the optimal public provision of liquidity in times of crisis. Following Walter Bagehot (Bagehot, 1873), the standard prescription for the central bank is to provide emergency liquidity 1 against penalty rates – in order to mitigate moral hazard. In our model, penalty rates are only optimal in specific circumstances; that is, when the cost of ex-post liquidity is sufficiently high. When this is not the case, low interest rates on public liquidity during crisis are optimal. This is because such interest rates deter overaccumulation of liquidity at banks during normal times and lead to a more efficient (ex-ante) allocation of liquidity between banks and the rest of the economy.

The next section of the paper describes the model. Sections 3 and 4 solve for the efficient and the private allocation of liquidity. In Section 5 we introduce public provision of liquidity. Section 6 endogenizes the ex-ante and ex-post liquidity friction. The final section concludes.

2. Model

The economy has three dates, \( t = 0, 1, 2 \). There are two types of agents, banks and investors. Both are risk neutral and consume at date 2. Bankers (indexed with \( j \)) come from a continuum of measure 1 and are each endowed with one project that returns \( R \) at date 2. Investors (indexed with \( i \)) come from a continuum of measure 1 and are endowed with one unit of wealth at date 0 (throughout the paper we will use subscripts \( i \) and \( j \) to denote individual quantities; omitting the subscript will indicate the corresponding economy-wide aggregates.)

There exists a storage technology (available to everybody) which turns one unit of wealth at date 1 into one unit of wealth at date 1 + 1. Our modeling of liquidity demand follows the framework of Holmström and Tirole (1998). Projects require no outside funding at \( t = 0 \), but may be hit by a (system-wide) liquidity shock at \( t = 1 \): with probability \( \pi (\pi \in (0, 1)) \) projects require a liquidity injection of \( \lambda > 0 \), which is returned at \( t = 2 \). \(^6\) If the liquidity is not provided, the project becomes worthless. There also exists a technology for (partial) liquidation of projects at \( t = 1 \). This technology turns \( 1 + \gamma \) units of date-2 project return into 1 unit at date 1. The liquidation cost \( \gamma \) depends on the total amount liquidated in the economy, denoted by \( l \). We assume that \( \gamma (l) > 0 \) and \( \gamma (l) \geq 0 \). Increasing costs can be thought of as arising from fire-sales of projects or other types of liquidation externalities (in Section 4.1 we explicitly model the cost as coming from outsiders who require increasing discounts when purchasing more assets).

The ability of bankers to raise funds from investors is subject to frictions. First, only a fraction \( \alpha (\alpha \in (0, 1)) \) of the date-2 return of the project can be pledged to outsiders. Limited pledgability may for example arise because bankers have project-specific skills that allow them to renegotiate promises to outsiders (as in Hart and Moore (1994)). Second, there is a cost to transferring funds from investors to banks. Specifically, in order for a banker to have one unit of funds at \( t = 1 \), he needs to raise \( 1 + \delta (\delta \in (0, 1)) \) funds from investors at \( t = 0 \). Similarly, there is a cost of transferring funds at \( t = 1 \), \( \delta (\delta \in (0, 1)) \). We assume that the liquidation technology is inferior to raising funds from investors:

\[ \text{Assumption 1} \quad \gamma (0) > \max (\delta \delta, \delta \delta) \]

We allow for \( \delta = \delta \), as ex-ante and ex-post costs to liquidity are likely to differ. For example, liquidity raised ex-post may be subject to higher adverse selection costs. Ex-ante liquidity, by contrast, may suffer from agency problems. In particular, a banker may pay himself some of the free funds before \( t = 1 \) through bonuses etc, or he may convert free funds into illiquid assets that are not pledgeable at \( t = 1 \) ("The paradox of liquidity", see Myers and Rajan, 1998). To capture this, we assume that a banker enjoys (non-monetary) benefits \( \phi \phi \) per unit of ex-ante

\[^6\] We focus on externalities arising from the asset side of banks; however, externalities can also arise on the liability side (see Farhi and Tirole, 2012; Stein, 2012; Segura and Suarez, 2017; Dewatripont and Tirole, 2018). In particular, in Farhi and Tirole (2012) the scope for liquidity regulation arises because banks have incentives to take on concentrated liquidity risks in order to benefit from systemic bailouts. In Dewatripont and Tirole (2018) a need for liquidity regulation arises due to an externality on public finance arising from (non-systemic) bailouts.

\[^5\] See Repullo (2000, 2005) and Castiglione and Wagner (2012) for other reasons why charging penalty rates may not be optimal.

\[^4\] We focus the analysis on the ex-ante/ex-post liquidity trade-off, and hence take the number of projects undertaken as fixed (the number of completed projects is endogenous though, due to possible liquidations at date 1). Extending the model to consider variable projects delivers fairly standard results (banks will start more illiquid projects than socially desirable).

\[^6\] The projects, and the resulting liquidity shocks, can be interpreted as relating to (unmodeled) firms that are funded by banks. Examples for such liquidity shocks are the GFC (during which firms massively drew on their credit lines at banks) and, more recently, the Repo-turmoil of September 2019 (which was at least partly caused by high liquidity demands from firms due to tax payments).

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1 There are also reasons why banks may hold excessive liquidity, arising from speculative motives or market power (see Acharya and Yorulmazer, 2011; Diamond and Rajan, 2011; Acharya et al., 2012).

2 We focus on externalities arising from the asset side of banks; however, externalities can also arise on the liability side (see Farhi and Tirole, 2012; Stein, 2012; Segura and Suarez, 2017; Dewatripont and Tirole, 2018). In particular, in Farhi and Tirole (2012) the scope for liquidity regulation arises because banks have incentives to take on (correlated) liquidity risks in order to benefit from systemic bailouts. In Dewatripont and Tirole (2018) a need for liquidity regulation arises due to an externality on public finance arising from (non-systemic) bailouts.

3 More generally, a rationale for central bank intervention may also arise because of co-ordination problems in the interbank market (Freixas et al., 2000), price indeterminacy (Allen et al., 2009) or because it can facilitate a better distribution of liquidity among banks (Freixas et al., 2011).

4 See Repullo (2000, 2005) and Castiglione and Wagner (2012) for other reasons why charging penalty rates may not be optimal.
liquidity. In Section 6 we discuss sources of cost differences in more detail and suggest that the relative size of ex-ante to ex-post cost in practice is likely to depend on the type of financial intermediary considered.

3. Efficient allocation

We consider a social planner who maximizes (utilitarian) welfare subject to a participation constraint. The constraint is that investors cannot be made worse off compared to autarky. Welfare is given by total expected date-2 consumption in the economy, plus any private costs accruing to bankers. If there were neither liquidations, funding costs nor private benefits (i.e., \( I = 0, \delta_1 = 0 \) and \( \phi_1 = 0 \)), welfare would simply be given by the sum of the date-2 project returns and investors' endowments: \( R + I \). However, in our analysis welfare will differ because of (i) losses from project liquidations at \( t = 1 \), (ii) the cost of transferring funds to bankers at \( t = 0 \) and \( t = 1 \), and (iii) private benefits \( \phi_0 \).

We denote by \( y_0 \) the amount of liquidity transferred from investors to banks at \( t = 0 \) (net of the cost \( \delta_0 \)) and by \( y_1 \) (the net) amount transferred at date 1 when the shock hits (when there is no shock, there is no need for liquidity transfer). Welfare can then be expressed as

\[
W(y_0, y_1, l) = R + I - (\delta_0 - \phi_0)y_0 - \pi\delta_1 y_1 - \pi\Gamma(l),
\]

(1)

where \( \Gamma(l) = \gamma(l) \) denotes total liquidation costs. The term \(- (\delta_0 - \phi_0)y_0\) is the loss from transferring liquidity at \( t = 0 \) (net of private benefits) which is incurred with certainty. The term \(- \pi\delta_1 y_1\) is the expected cost of transferring liquidity at \( t = 1 \), materializing when the shock arrives. The last term, \(- \pi\Gamma(l)\), is the expected liquidation costs if, when the shock arrives, \( l \) unit of assets are liquidated at a cost of \( \gamma(l) \) each.

The social planner faces the following constraints. First, the date-1 liquidity at banks (when the shock hits) has to be at least as high as the liquidity need \( \lambda \). Given liquidity holdings from date 0, \( y_0 \), new liquidity transferred, \( y_1 \), and liquidation proceeds, \( l \), this condition can be written:

\[
y_0 + y_1 + l \geq \lambda.
\]

(2)

Second, the liquidity transferred to the banks cannot exceed investors' endowments:

\[
(1 + \delta_0)y_0 + (1 + \delta_1)y_1 \leq l.
\]

(3)

Third, the borrowing capacity of bankers is restricted. This means that the total amount of date-2 funds promised at date 0 and date 1 cannot exceed the pledgeable return \( aR \). Given that investors have to earn at least the autarky return, this condition is

\[
(1 + \delta_0)y_0 + (1 + \delta_1)y_1 \leq aR.
\]

(4)

The planner's problem can therefore be stated as

\[
\max_{y_0, y_1, l} W(y_0, y_1, l), \quad \text{subject to } (2), (3), (6) \text{ and } y_0, y_1, l \geq 0.
\]

(5)

We turn to the solution of the problem. We restrict attention to parameter values for which when the liquidity shock hits, at least one of the liquidity constraints (the supply constraint (3) or the borrowing capacity constraint (4)) binds. A sufficient condition for this is that the liquidity shock \( \lambda \) exceeds the total amount banks can raise from investors if they use the cheapest liquidity channel:

\[\text{Assumption 2 } \lambda - \frac{H}{1 + \min[\delta_0, \delta_1]} > 0,\]

where \( H = \min[I, aR] \).

When the shock hits at \( t = 1 \), liquidity holdings at banks should never exceed the liquidity requirement; otherwise, welfare could be improved by raising less at \( t = 0 \) or at \( t = 1 \) (and hence saving on liquidity cost \( \delta_0 \) or \( \delta_1 \)) or by liquidating less (and saving on the cost \( \gamma \)). Thus equation (2) holds with equality

\[
y_0 + y_1 + l = \lambda.
\]

(6)

How much liquidity is ex-post raised from investors, \( y_1 \), and how much from liquidations, \( l \)? Consider first the case that the supply constraint is the binding one: \( I \leq aR \). The supply constraint, equation (3), holds then with equality. Solving it for \( y_1 \) we obtain

\[
y_1 = \frac{H - (1 + \delta_0)y_0}{1 + \delta_1}.
\]

(7)

The remaining liquidity is generated through liquidations. Using (6) and (7) we obtain

\[
l = \lambda - \frac{H + (\delta_1 - \delta_0)y_0}{1 + \delta_1}.
\]

(8)

Consider next that the borrowing constraint is the binding one: \( I > aR \). In this case, liquidity will be transferred to bankers up to their borrowing capacity; the remaining liquidity will again be generated through liquidations. It is optimal to undertake such liquidations on the project’s non-pledgeable part in order not to reduce the borrowing capacity. A condition that guarantees that the non-pledgeable part is large enough to generate the required liquidity is given by:

\[\text{Assumption 3 } \left(1 + \gamma \left( \lambda - \frac{H}{1 + \min[\delta_0, \delta_1]} \right) \right) \left(1 - \frac{H}{1 + \min[\delta_0, \delta_1]} \right) \leq (1 - \alpha)R\]

(9)

Liquidation transfers \( y_1 \) and liquidations \( l \) are then still given by equations (7) and (8).

At \( t = 0 \), the social planner has to decide upon how much liquidity, \( y_0 \), to transfer to bankers. Substituting \( y_1 \) and \( l \) in equation (1), we obtain welfare as a function of \( y_0 \) only:

\[
W(y_0) = R + I - (\delta_0 - \phi_0)y_0 - \pi\delta_1 y_1 - \pi\Gamma(l(y_0)).
\]

(10)

What determines optimal holdings of precautionary liquidity? By differentiating (9) with respect to \( y_0 \) we obtain the (marginal) benefits of precautionary liquidity:

\[
W'(y_0) = - (\delta_0 - \phi_0) - \alpha\delta_1 y_1^* - \pi\Gamma'(l(y_0)).
\]

(11)

The first two terms represent the trade-off that arises due to the cost of investor liquidity. More ex-ante liquidity means that costs of \( \delta_0 - \phi_0 \) are incurred at date 0. This provides benefits ex-post when the liquidity shock arises as less liquidity then has to be transferred (\( y_1^*(y_0) < 0 \)) and hence less of the \( \delta_1 \) cost is incurred. Precareutionary liquidity thus trades off incurred \( \delta_0 - \phi_0 \) at \( t = 0 \) for sure against incurring \( \delta_1 \) at \( t = 1 \) with probability \( \alpha \).

The last term is the effect on the liquidation costs and depends on the relative size of \( \delta_1 \) and \( \delta_0 \) (we have that \( \text{sign} (\Gamma'(l(y_0))) = - \text{sign} (\delta_1 - \delta_0) \)). What is the reason for this? Consider first the case of the supply constraint. When \( \delta_0 < \delta_1 \), ex-ante transfers are more effective than ex-post transfers in the sense that they maximize the amount of investor liquidity that can be made available to banks at \( t = 1 \). They thus lower liquidations and hence liquidation costs. When \( \delta_1 < \delta_0 \) exactly the opposite argument arises and ex-post liquidity lowers liquidations. The case of the borrowing constraint follows the same logic. This constraint limits the amount that can be pledged to

\[\text{...} \]
outsiders, which translates into a maximum amount of liquidity that can be transferred to banks. So using the more effective means of liquidity alleviates the borrowing constraints and lowers liquidations.8

Optimal liquidity holdings thus minimize the cost from transferring liquidity to banks and the amount of liquidations. A trade-off arises when one channel of liquidity is more costly (in terms of the liquidity costs δ) but makes liquidations less likely. For example, when δ1 - φ1 > nR(1,yj), and δ2 < δ1, ex-ante liquidity is more costly but reduces liquidations. To see the effects more precisely, the following proposition examines the comparative statics of interior solutions when the (aggregate) liquidation cost function is linear:

**Proposition 1.** For linear liquidation costs γ(l) = yj + μl (l > 0) efficient (interior) precautionary liquidity yj⁰

(i) increases (decreases) in the size of the liquidity shock λ when δ1 > δ0 (δ1 < δ0);
(ii) decreases (increases) in the size of the investor sector I and the pledgeable return aR when δ1 > δ0 (δ1 < δ0);
(iii) increases (decreases) in the cost parameters γ0 and γ1 when δ1 > δ0 (δ1 < δ0);
(iv) increases in the private benefits φj.

**Proof.** See appendix. □

The results (i)-(iii) all follow the same intuition. A parameter change that increases the liquidation problem (an increase in the shock λ, a tightening of the liquidation constraints through a reduction in a or aR, or an increase in the costs 𝛾, γ1 ) will increase (unit) liquidation costs γ. This, in turn, makes avoiding liquidations more beneficial. The consequence of this will be that the more effective channel of liquidity becomes more valuable, making it optimal to hold more ex-ante liquidity when δ1 < δ1 and more ex-post liquidity when δ1 > δ1. Part (iv) of the proposition shows, as to be expected, that higher private benefits from ex-ante liquidity make holding more of it optimal.

4. Equilibrium

We start by deriving the optimization problems for bankers and investors. A banker has to decide how much precautionary liquidity to raise from investors (yj₀). At t = 1, when the liquidity shock hits, he also has to decide how much liquidity to raise ex-post (yj₁) and how much to liquidate (l). In his optimization, the banker takes as given the interest rate on (two-period) borrowing from investors at date 0, r₀, the interest rate at date 1 when the liquidity shock hits, r₁, as well as the liquidation cost γ.

A banker faces two constraints, which are similar to the one of the social planner. First, when the shock materializes, his liquidity (after borrowing and liquidations) has to be at least as high as the liquidity need λ:

yj₀ + yj₁ + l ≥ λ. (11)

Second, his borrowing capacity cannot be exceeded:

(1 + δ₀)(1 + r₀)yj₀ + (1 + δ₁)(1 + r₁)yj₁ ≤ qR. (12)

The expected utility for the banker can be written as the project return R, minus costs arising from ex-ante and ex-post liquidity, as well as liquidations:

W⁽¹⁾(yj₀, yj₁, l) = R - ((1 + δ₀)(1 + r₀) - φ₀ - 1)yj₀

- π((1 + δ₁)(1 + r₁) - φ₁ - 1)yj₁ - πγ(l). (13)

The term ((1 + δ₀)(1 + r₀) - φ₀ - 1)yj₀ is the cost of precautionary liquidity, arising because the liquidity cost is higher than the private benefit and because interest has to be paid. The term - π((1 + δ₁)(1 + r₁) - φ₁ - 1)yj₁ is the expected cost of ex-post liquidity, due to liquidity costs and interests. The final term - πγ(l) is the banker’s expected cost due to liquidations. This cost depends on both the banker’s liquidations l and aggregate liquidations l as the latter determine the unit costs γ.

Investors have to decide how much to lend at t = 0 and when the shock hits at t = 1. They will store any unused liquidity for the next period. Denoting lending at t = 0 and t = 1 with s₀ and s₁, respectively, an investor’s expected utility can be written as:

W⁽²⁾(s₀, s₁) = 1 + r₀s₀ + πγs₁. (14)

**Definition 1.** A competitive equilibrium consists of interest rates r⁰ and r₁, bank liquidity and liquidation choices (yj₀, yj₁, l) in eq(1,1) and investor lending choices (s₀, s₁) in eq(0,1) such that

(i) for each bank j: yj₀, yj₁, l maximize expected profits W⁽¹⁾ subject to (11), (12) and yj₀, yj₁, l ≥ 0;
(ii) for each investor j: s₀, s₁ maximize expected profits W⁽²⁾, subject to s₀, s₁ ≥ 0 and s₀ + s₁ ≤ 1;
(iii) the markets for liquidity clear at t = 0 and t = 1: (1 + δ₀)yj₀ = s₀

and (1 + δ₁)yj₁ = s₁.

**Lemma 1.** At an equilibrium, the difference between a banker’s and the social (marginal) benefits of precautionary liquidity is given by

W⁽¹⁾(yj₀) - W⁽¹⁾(yj₁) = πγ(l)l'(yj₀)l(yj₁). (15)

**Proof.** See appendix. □

Private and social benefits from liquidity differ due to an externality. An individual banker does not take into account that his liquidity decision affects aggregate liquidations l and hence the unit liquidation costs γ for all other banks in the economy. The sign of the externality depends again on the relative costs. In particular, when δ₀ < δ₁, precautionary liquidity reduces liquidations (l'(yj₀) < 0) and hence creates a positive externality from precautionary liquidity, while when δ₀ > δ₁, the externality is a negative one.

We focus in what follows on economies in which both efficient and equilibrium (precautionary) liquidity are interior. The following proposition derives then the efficiency properties of the equilibrium.10

**Proposition 2.** The equilibrium amount of precautionary liquidity yj₀ differs from the (constrained) efficient one yj₀⁽¹⁾ whenever δ₀ ≠ δ₁. In particular:

(i) When δ₀ < δ₁, equilibrium liquidity is less than the (constrained) efficient one: yj₀ < yj₀⁽¹⁾;
(ii) When δ₀ > δ₁, equilibrium liquidity exceeds the (constrained) efficient one: yj₀ > yj₀⁽¹⁾.

**Note.** That in equilibrium only aggregate precautionary liquidity, yj₀, is determined – but not the individual holdings at each bank (each bank is indifferent to the amount of own precautionary liquidity).

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8 Whereas in our analysis the source of (costly) liquidations are liquidity shortages, a large part of the literature has emphasized runs on (illiquid) banks arising from coordination problems among depositors (e.g., Diamond and Dybvig, 1983; Calomiris and Kahn, 1991). We would expect ex-ante and ex-post liquidity to play a similar role in the presence of such runs. This is because using the more effective source of liquidity will increase the ability of banks to stave off of liquidity-based runs.

9 There is no scope for interbank borrowing in our analysis as banks have identical risk exposures. However, the model can also be interpreted as (reduced-form) one in which banks are differentially exposed to shocks and the interbank market first (perfectly) smooths out differences in liquidity needs across banks.

10 Note that in equilibrium only aggregate precautionary liquidity, yj₀, is determined – but not the individual holdings at each bank (each bank is indifferent to the amount of own precautionary liquidity).
Proof. See appendix. □

The inefficiency depends on the relative size of \( \delta_0 \) and \( \delta_1 \), regardless of which liquidity constraint binds. In the case of the supply constraint, using the more effective channel for raising liquidity (that is, ex-ante liquidity when \( \delta_0 \leq \delta_1 \) and ex-post liquidity when \( \delta_1 > \delta_0 \)) increases the total liquidity available to banks at \( t = 1 \). This in turn lowers liquidations and liquidation costs, with positive effects for all liquidating banks. There is thus a positive externality associated with using the (privately) most efficient way of raising liquidity. In the case of the borrowing constraint, using the more effective channel means that for given borrowing capacity banks maximize the amount of liquidity they can raise without exceeding the constraint, again lowering liquidations and unit liquidation costs. However, the level of aggregation at which the externality operates differs. In the case of constrained liquidity supply, individual liquidity holdings at \( t = 0 \) affect aggregate liquidity available at \( t = 1 \), and through this affect liquidations. In the case of borrowing constraints, individual liquidity holdings at a bank affect all banks’ individual constraints at \( t = 1 \), and through this liquidations.

The following corollary summarizes the welfare implications for standard liquidity requirements, showing that they can be welfare reducing.\(^{11}\)

**Corollary 1.** A minimum liquidity requirement \( e_i \) can implement (constrained) efficient precautionary liquidity whenever \( \delta_0 \leq \delta_1 \). Whenever \( \delta_1 > \delta_0 \), any binding liquidity requirement (i.e., \( e_i \geq e_i^{\text{opt}} \)) reduces welfare.

**Proof.** Follows directly from Proposition 2. When \( \delta_0 \leq \delta_1 \), setting \( e_i = e_i^{\text{opt}} \) will implement efficiency. When \( \delta_1 > \delta_0 \), equilibrium liquidity exceeds the efficient one \( (e_i^{\text{opt}} > e_i^{\text{opt}}) \). Any liquidity requirement set in excess of \( e_i^{\text{opt}} \) will thus lower efficiency. □

**4.1. Asset sales and liquidation costs**

We have assumed liquidation costs that are increasing in the total amount of liquidations in the economy. One interpretation of this is asset sales. When liquidations take the form of asset sales, higher selling in the economy results in lower prices (“fire-sale” prices), thus increasing liquidation costs for all banks liquidating. In Appendix B we model this channel explicitly. We consider non-bank consumers who may purchase assets but require a discount for doing so. The compensation they require for taking on assets increases in the total amount of asset already purchased because of risk-aversion: the marginal compensation for a unit of risk increases when the buyer is already exposed to this risk. We show that this results in an increasing liquidation cost function and that, as a result, the same wedge as in Proposition 2 arises between efficient and equilibrium liquidity holdings.

**5. Public provision of liquidity**

In this section we introduce liquidity supply by a central bank (CB). Public liquidity has two potential functions our model. First, it may lower liquidity shortages at \( t = 1 \) and reduce costly liquidations.\(^{12}\) Second, it may change the equilibrium allocation of liquidity in the economy—this may be desirable since this allocation is inefficient when \( \delta_0 = \delta_1 \) (Proposition 2).

We model the CB as follows. The CB has a total stock of liquidity \( m \).\(^{13}\) The CB can lend this stock to banks by setting up borrowing facilities at date 0 and date 1. A borrowing facility is characterized by its interest rate \( r_i^{CB} \) and the size \( m_i \). The CB’s objective is to choose these parameters such that welfare is maximized. CB liquidity is subject to the same date-0 and date-1 frictions as liquidity from investors. To focus on (pure) liquidity policies, we assume that the CB cannot lend at an expected return that is less than the return on storage \( (r_i^{CB} \geq 0) \). Finally, to avoid ambiguity, when different liquidity policies obtain the same level of welfare, we assume that the CB chooses the one that results in the lowest expected amount of CB liquidity injected into the economy (this can be justified by a small deadweight cost of providing liquidity).

We denote with \( y_i \) the amount of private liquidity (investor liquidity) borrowed by banks, with \( y_i^{CB} \) the public (CB) liquidity borrowed and with \( y_i = y_i^{CB} + y_i^{CB} \) total liquidity borrowed at \( t \). Welfare is identical to equation (1) amended for the stock of public liquidity \( m_i \). Denoting \( f_i = l + m_i \) we have

\[
W = R + \bar{I} - (\delta_i - \theta_i)y_i - \pi\delta_i y_i - \pi\bar{I}(l).
\]

(16)

The new market clearing conditions for liquidity are given by

\[
(1 + \delta_i)y_i = s_i + m_i.
\]

(17)

A banker’s date-1 borrowing constraint is determined by the sum of private and public liquidity raised and is given by

\[
(1 + \delta_i)(1 + r_i)y_i^l + (1 + \delta_i)(1 + r_i^{CB})y_i^{CB} + (1 + \delta_i)(1 + r_i)y_i^t
\]

\[
\geq \alpha R
\]

and his utility is

\[
W_i^{CB} = R - ((1 + \delta_i)(1 + r_i) - \theta_i - 1)y_i^t - (1 + \delta_i)(1 + r_i^{CB}) - 1)y_i^{CB}
\]

\[\geq - \pi((1 + \delta_i)(1 + r_i) - 1)y_i^t + ((1 + \delta_i)(1 + r_i^{CB}) - 1)y_i^{CB}
\]

\[\geq - \pi\bar{I}(l_i).
\]

(18)

We obtain the following modified definition of an equilibrium:

**Definition 2.** An equilibrium with CB lending \( (m_i, r_i^{CB}, r_i, y_i^{CB}) \) consists of interest rates \( r_i^{CB} \) and \( r_i \), bank liquidity and loaning choices \((y_i^{CB}, y_i^{CB}, y_i^{CB}, y_i^{CB}, y_i^{CB}) \in [0,1] \) and investor lending choices \((s_i, s_i) \in [0,1] \) such that

(i) for each bank \( f_i^{l} \), \( y_i^{CB}, y_i^{CB}, y_i^{CB}, y_i^{CB}, y_i^{CB} \) maximize expected profits \( W_i^{CB} \) subject to (11), (19) and \( y_i^{CB}, y_i^{CB}, y_i^{CB}, y_i^{CB}, y_i^{CB}, b_i \geq 0 \);

(ii) for each investor \( s_i, s_i \) maximize expected profits \( W_i \) subject to \( s_i + s_i \geq 0 \) and \( s_i + s_i \leq 1 \);

(iii) the market for liquidity clears at \( t = 0 \) and \( t = 1 \):

\[
(1 + \delta_i)y_i^t = s_i + m_i \text{ and } (1 + \delta_i)y_i^t = s_i + m_i;
\]

(iv) the borrowing facility is not exceeded at \( t = 0 \) and \( t = 1 \);

(footnote continued)

prevent panic runs at illiquid banks (e.g., following the work of Bagehot, 1873). By contrast, in our model the function of (crisis) liquidity interventions is to alleviate structural liquidity shortages. For example, during the GFC central banks around the world introduced measures to reduce liquidity deficits in the financial system.

\(^{13}\) Limits to (public) liquidity provision are a tractable way to introduce the notion that central bank liquidity is not costless (in particular: it can cause inflation) and/or that the central bank lacks credibility. They have the consequence of making CB liquidity scarce, forcing the CB to economize on its liquidity provision. An alternative to liquidity limits is to model social costs of liquidity provision explicitly (see, for example, Dewatripont and Tirole, 2018).
\[ y_{1t}^{CB} \leq m_0 \text{ and } y_{1t}^{CB} \leq m_1. \]

The equilibrium conditions for banks and investors ((i) and (ii)) are unchanged – except for banks who can now also borrow from the CB. Condition (iii) is the new market clearing for liquidity at \( t = 0 \) and \( t = 1 \), whereas condition (iv) states that banks cannot borrow more from the CB than the CB offers them.

An analysis similar to that in Section 4 can be used to solve for the equilibrium with CB lending. This equilibrium outcome can then be used to determine welfare \( W^* \); it will be a function of the CB policy choice \((m_0, r_0^m, m_1, r_1^m)\). The CB’s problem is then to set the parameters of the borrowing facilities such that the equilibrium with the highest welfare value:

\[
\max_{m_0, \delta_0, \delta_1} W^* \text{, subject to } m_0 + m_1 \leq m \text{ and } r_0^m, r_1^m \geq 0. \tag{21}
\]

Proposition 3 summarizes the efficiency implications of public liquidity provision (the optimal policies themselves are derived in the proof in the appendix).

**Proposition 3.** Liquidity policies can implement the (constrained) efficient solution unless the borrowing constraint binds \((\alpha < 0 \text{ and } \lambda - \frac{H}{\rho(\delta_0^*, \delta_1^*)} > 0). In the latter case, precautionary liquidity will remain insufficient (excessive) under an optimal liquidity policy when \( \delta_0 < \delta_1^*(\delta_0 > \delta_1^*). \)

**Proof.** See appendix. □

The reason why liquidity policies can implement efficiency when the liquidity constraint binds (but not the borrowing constraint) binds as is follows. When the liquidity constraint binds, the date-1 market interest rate (in absence of CB lending) is positive. By setting a sufficiently low price on liquidity (relative to the pre-CB market rate), the CB can fully control liquidity take-up at banks. It can hence control the private allocation of liquidity, and correct the liquidity bias in private allocations. If it is the borrowing constraint which binds, the (market) interest rate (prior to CB lending) is already zero. The CB is then unable to influence the price of liquidity in the economy without lending at a loss. As a result, the CB cannot influence the liquidity positions of banks, and the efficiency bias (Proposition 2) remains.\(^{14}\)

This analysis has direct implications for whether a CB should pursue active ex-ante policies. Again, the scope for such a policy depends on the type of shortage. If a banking system tends to experience problems because of liquidity supply problems, there is a clear role for setting interest rates. The optimal policy requires “to lean against” the bias created by the externalties (see the proof of Proposition 3). When \( \delta_0 < \delta_1^* \), this implies that the CB may want to stimulate precautionary liquidity by reducing interest rates in normal times (ex-ante). By contrast, when \( \delta_0 > \delta_1^* \), the CB should try to drain liquidity from the banking system. It can do so by keeping ex-ante interest rates high to discourage borrowing, while at the same time promising low interest rates in crises (which increases the benefits from relying on ex-post liquidity).\(^{15}\) However, if a banking system tends to experience problems because of a limited ability to borrow, there is no role for influencing ex-ante interest rates. This is for the same reason as why the CB then cannot influence the liquidity positions (under the assumption of no bailout).

## 6. Liquidity frictions

In this section we provide illustrative examples of the ex-ante and ex-post liquidity cost, \( \delta_0^* \) and \( \delta_1^* \), and discuss what determines their relative size in practice.

Let us start with an explanation of the general nature of the liquidity frictions in our analysis. The literature has predominantly modelled frictions that take the form of limits to raising liquidity (for example, pledgeability constraints as in Holmström and Tirole, 1998). By contrast, the ex-ante and ex-post frictions in our analysis take the form of costs that reduce total liquidity available for satisfying liquidity needs. The ex-ante friction is any type of inefficiency that lowers the availability of funds that have been supplied by investors to banks at \( t = 0 \) for satisfying liquidity needs at \( t = 1 \). This could be a direct friction when raising the funds at date 0 (for example, a deadweight cost to raising capital), or something that happens between date 0 and date 1. The ex-post friction is any type of inefficiency that (during an aggregate crisis) creates a wedge between the amount of funds supplied by investors to banks at \( t = 1 \), and the amount of funds that can use to satisfy liquidity needs. These ex-ante and ex-post frictions then interact with traditional limits to liquidity. For example, for a given limit on funds a bank can raise in total ex-ante and ex-post (coming, for instance, from a pledgeability constraint), “destroying” liquidity through ex-ante or ex-post costs will lower banks’ ability to satisfy liquidity needs.

We modify the setup in Section 2 as follows. At date 0, banks raise liquidity from investors without incurring any cost at first. Banks are run by a manager who can divert a fraction \( \phi_0^*/\phi_0 \) of the (free) liquidity between date 0 and date 1, providing the manager with private benefits \( \phi_0^*/\phi_0 \) per unit of appropriated funds. This may take the form of managers converting funds into illiquid assets that are not pledgeable at date 1, or paying themselves bonuses. It follows that in order for a bank to arrive at date 1 with \( y_0^* \) units of liquidity, the bank has to raise \((1 + \delta_0^*)y_0^* \) units of funds from investors at date 0. Expressed in terms of (net) liquidity \( y_0^* \), the manager enjoys private benefits of \( \phi_0^*/\phi_0 \).

In addition, we assume that at date 1 a mass of \( \delta_1^* \) “bad” banks appear who pose as ordinary banks. Bad banks operate worthless projects: they require an input of \( j \) at \( t = 1 \) but only provide private benefits \( \phi_1^* \) of the (free) liquidity between date 0 and date 1, providing the manager with private benefits \( \phi_1^*/\phi_1 \) per unit of appropriated funds. This may take the form of managers converting funds into illiquid assets that are not pledgeable at date 1, or paying themselves bonuses. It follows that in order for a bank to arrive at date 1 with \( y_0^* \) units of liquidity, the bank has to raise \((1 + \delta_1^*)y_0^* \) units of funds from investors at date 0. Expressed in terms of (net) liquidity \( y_0^* \), the manager enjoys private benefits of \( \phi_0^*/\phi_0 \).

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Our modelling of ex-ante and ex-post frictions aims to capture that in “crisis times” (when banks are hit by negative shocks) it is particularly difficult for outsiders to make sure they invest in viable banks with good projects. Uncertainty about asset quality is likely to be a major obstacle to raising funds in such situations.\(^{16}\) By contrast, in “normal

---

\(^{14}\)If we also allow the CB to lend at a negative rate, the allocation can be improved along two dimensions. First, the CB (or the government) can then provide liquidity to banks without requiring repayment. This relaxes the borrowing constraint, as CB liquidity is no longer included in the constraint, leading to a lower amount of liquidations in the economy. Second, lending at a negative rate gives the CB the power to replace the market at either \( t = 0 \) and \( t = 1 \). Thus the CB has full control over liquidity in the economy; it can hence correct any misallocation that occurs when \( \delta_0 = \delta_1^* \).

\(^{15}\)In Diamond and Rajan (2012) the CB charges high rates (in normal times) in order to curb moral hazard arising from cheap emergency funding during crisis. The motive for setting high interest rates in our context is different. It arises because banks may hold excessive ex-ante liquidity, rather than to offset moral hazard arising from (anticipated) liquidity policies.

\(^{16}\)This feature is also key in the model of Bolton et al. (2011), where
times” (date 0 in our model), concerns about asset quality are less paramount. In such times, inefficiencies due to agency problems within the bank are probably of higher relevance.

It is easy to show (calculations available on request) that in the case of unsophisticated investors the model is identical to the one of Section 2. Proposition 2 still holds: there is a bias in liquidity holdings – with the direction of the bias depending on the relative size of ex-ante and ex-post costs. The case of sophisticated investors is also straightforward. These investors have no problems discerning good banks when lending at t = 1. The presence of bad banks is hence irrelevant. The problem is thus identical to the case of unsophisticated investors and no bad banks (δ1 = 0). From Proposition 2 we then have that sophisticated investors will lend inefficiently high amounts to banks ex-ante.

6.1. Discussion

The analysis in this section helps us to understand for which types of financial institutions – in practice – we would expect precautionary liquidity to be too low or too high. 17 First, the direction of the liquidity externality will depend on the characteristics of the investors an institution deals with. Consider for example a small traditional commercial bank, financing itself with retail deposits. This bank largely deals with unsophisticated investors. For retail investors the cost of providing liquidity in times of crisis when informational asymmetries are pronounced is high. We are thus likely to have the case of high δ1 relative to δ0 and precautionary liquidity is probably too low. Consider, alternatively, a bank that finances itself in the wholesale funding market. Its funding base may then come to a larger share from informed investors who are better able to deal with asset quality problems that are key in a crisis. 18 The relative costs of raising funds ex-post from these investors is thus low (δ1 is small) and pre-cautionary liquidity holdings may be excessive.

Another application is relationship lending to banks. Consider an institution that has an established relationship with a lender (for example, through previous borrowing activities). This lender is then fairly informed about the bank and is hence able to continue funding the bank in a crisis (this is basically the same argument as for relationship lending to firms). Liquidity provision by such a lender is thus likely to be characterized by low relative ex-post costs. We would thus expect that banks that have access to “relationship” funding to overutilize this channel of liquidity (prior to crisis). 19 Our analysis also suggests that the direction of the liquidity externality depends on the scope for asset substitution by managers (proxied by δ0 in the model). For institutions where managers have large discretion and where there is little market discipline, it is wasteful to have spare liquidity (as this would lead to a significant reduction in the liquidity that is available to banks in crises); instead liquidity should only be raised when liquidity problems materialize. The potential for asset substitution is more significant at institutions that engage in opaque activities – such as proprietary trading and investment in OTC derivatives. This would point to large and complex banks as well as investment banks as institutions where precautionary liquidity may be inefficiently high.

We can also expect the liquidity externality to depend on the frequency with which liquidity shortages occur. If liquidity problems do not arise often, a longer period of time will (on average) pass between the raising of precautionary liquidity and the arrival of a liquidity need. This would worsen the asset substitution problem as managers have more time to divert funds (δ0 is high), suggesting a higher potential for a negative liquidity externality. 20 We would thus expect institutions that encounter liquidity shortages infrequently to raise an inefficiently high amount of liquidity ex-ante, while institutions that regularly experience significant liquidity problems may underinvest in liquidity.

Finally, the direction of the liquidity externality also depends on how one interprets the time scale. If liquidity is raised at the time of the initial investment in projects, potential future liquidity needs may be far ahead (and hence δ0 large). By contrast, if banks have the possibility to raise funds in direct anticipation of an impending liquidity shock, δ0 will be low (as in this case there is little time for the manager to misuse funds). This suggests that the potential for excessive precautionary liquidity arises with respect to liquidity that is raised in normal times and when there is no immediate threat of a shortage. By contrast, liquidity raised just before a potential shortage may be insufficient from the social perspective.

7. Summary and conclusions

We have developed a model in which banks can raise liquidity in anticipation of needs, but also once liquidity shocks materialize. A consequence of the latter possibility is that the intuition that banks tend to hold insufficient amounts of liquidity (due to its public good character) no longer holds, as banks can still raise liquidity once hit by a shock. In situations where ex-ante liquidity is subject to frictions, precautionary liquidity holdings can even undermine the banking sector’s ability to generate funds in crises.

Our model provides several insights. First, while the focus of the banking literature has mostly been on the optimal allocation of resources within the banking sector, our paper shows that inefficiencies can also arise across sectors. Banking policies (such as liquidity ratios or capital requirements) should hence take into account their impact on the cross-sector allocations of funds. Second, optimal liquidity policies may differ by type of financial institution. Precautionary liquidity holdings may have to be discouraged at institutions that have relatively high ex-ante costs from raising liquidity and at institutions that tend to raise liquidity from a confined pool of informed investors. At other institutions, there is a rationale for regulators to encourage precautionary liquidity holdings. Third, optimal liquidity policies may take the form of discouraging holding liquidity at banks, while encouraging the provision of liquidity to banks in crises times. This may be achieved by setting high interest rates in normal times, and low interest rates during crises.

Our analysis has abstracted from various issues. For one, we have taken as given banks’ investment in projects. Dispensing with this assumption will create an additional inefficiency (arising from excessive investment in illiquid projects) but does not modify the liquidity results.

(footnote continued)

17 Note that ex-ante and ex-posts are likely to be highly correlated. For example, institutions with weak governance (a proxy for ex-ante costs) may find it more difficult to raise funds during crises. Similarly, at institutions with opaque assets (a proxy for ex-post costs) managers may also find it easier to divert funds in normal times. Instead of attempting to find good proxies for ex-ante and ex-post costs separately, our approach in the following is to directly think about situations where we may expect ex-ante and ex-post costs to differ.

18 There is evidence that wholesale markets are more discriminatory during crises (that is, they fund good banks but not bad banks). For example, Périsignon et al. (2018) show that banks with high future performance see their share of wholesale funding increase during market stress (they also find that wholesale funding dry-ups for individual banks predict (but do not cause) future negative performance).

19 This argument implies that banks may “underinvest” in relationship funding. The reason is that increasing relationship funding increases the share of liquidity in the financial system that is informed, resulting in better overall liquidity allocations and lower liquidation externalities.

20 This could be viewed in our model as making the number of periods that pass until a liquidity shortage arises random and giving the manager the opportunity to appropriate a share of (uninvested) funds in every period.
Second, we have assumed that the amount of liquidity available in the investor sector in times of crisis falls one-to-one with the amount investors supplied to the banking sector in normal times. A more complete model of the world would consider intermediate consumption choices of investors and/or allow investors to make illiquid investments as well. The link between ex-ante and ex-post liquidity supply would then be no longer one-to-one (and possibly either smaller or larger) but our main insights should nonetheless continue to apply, since we can still expect a negative relationship between ex-ante and ex-post liquidity supply. Finally, our model focuses on liquidity needs that arise from banks. Investors, however, may also be subject to liquidity shocks, in which case the interaction between the two sectors becomes a two-way one. This is an interesting problem for future research.

Appendix A. Proofs

Proof of Proposition 1. We derive the comparative statics using the implicit function theorem (the model can also be solved explicitly for linear costs, however, tackling the problem more generally allows to better get the intuition behind the various effects). The first-order condition for efficient liquidity can be written as \( W'(y_0, r) = 0 \), where \( r \) represents one of the models parameters. Totally differentiating with respect to \( r \) and solving for \( y_0'(r) \) we obtain the comparative statics for \( r \):

\[
y_0'(r) = \frac{\frac{\partial W'(y_0)}{\partial r}}{-W'(y_0)}.
\]

(22)

For linear costs, \( W'(y_0) \) is given by (from equation (10)):

\[
W'(y_0) = -(\delta_0 - \phi_0) + \pi \frac{1 + \delta_0}{1 + \delta_1} + \pi (y_0 + \gamma_0 (\lambda - \frac{\min[I, \alpha R] + (\delta_0 - \delta_1)y_{eff}}{1 + \delta_1})) \frac{\delta_1 - \delta_0}{1 + \delta_1}.
\]

(23)

We have that the optimization problem is concave as the second derivative of welfare is negative:

\[
W''(y_0) = -\pi y_0 (\delta_1 - \delta_0)^2 < 0.
\]

(24)

It follows that at an interior solution the sign of \( y_0'(r) \) is given by the sign of \( \frac{\partial W'(y_0)}{\partial r} \). We focus on the comparative statics for which clear results can be obtained (for the ex-ante and ex-post costs as well as the the likelihood of a liquidity shock the comparative statics depend on the whole set of parameters).

Impact of the size of the investor sector \( I \): The size of the investor sector (and hence the total supply of liquidity in the economy) matters when the supply constraint is the binding one (\( I \leq \alpha R \)). From differentiating (23) with respect to \( I \) (assuming \( I \leq \alpha R \)), we obtain:

\[
\frac{\partial W'(y_0)}{\partial I} = -\pi y_0 (\delta_1 - \delta_0)^2.
\]

(25)

An increase in \( I \) will hence reduce efficient liquidity holdings \( y_{eff} \) when \( \delta_0 < \delta_1 \), and raise them when \( \delta_0 > \delta_1 \). What is the intuition for this result? A higher supply of liquidity by investors means that less assets have to be liquidated when the liquidity shock hits (that is, \( I \) falls). The unit-liquidation costs \( \gamma \) will hence be smaller. This has the consequence that the benefits from lowering liquidity shortages decline. As a result, it becomes less important to choose the more effective means of raising liquidity. In particular, when precautionary liquidity is the more effective one, its optimal amount of it falls.

Impact of the pledgeable return \( \alpha R \): This case matters when \( \alpha R > I \) and its analysis mirrors the prior one of \( I \leq \alpha R \). From differentiating (23) with respect to \( \alpha R \) (\( \alpha R > I \)), we obtain:

\[
\frac{\partial W'(y_0)}{\partial (\alpha R)} = -\pi y_0 (\delta_1 - \delta_0)^2.
\]

(26)

identical to (25). It follows that when the tightness of the liquidity constraint falls because of higher borrowing capacity, the more effective means of raising liquidity becomes less attractive.

Impact of the liquidity shock \( \lambda \):

The derivative with respect to \( \lambda \) is:

\[
\frac{\partial W'(y_0)}{\partial \lambda} = \pi y_0 (\delta_1 - \delta_0) \frac{\delta_1 - \delta_0}{1 + \delta_1}.
\]

(27)

The impact depends again on the relative costs, but with reversed sign. A higher liquidity demand at \( \lambda = 1 \) means, ceteris paribus, that unit liquidation costs will be higher as more assets are liquidated. This means that the more effective mean of raising liquidity becomes more valuable, hence \( y_{eff} \) increases when \( \delta_0 < \delta_1 \) and falls when \( \delta_0 > \delta_1 \).

Impact of the cost parameters \( \gamma_0 \) and \( \gamma_1 \):

A higher \( \gamma_0 \) or \( \gamma_1 \) raises the unit liquidation costs. Avoiding liquidations thus becomes more valuable, favouring again the more effective means of raising liquidity. This is confirmed by the relevant derivatives:

\[
\frac{\partial W'(y_0)}{\partial \gamma_0} = \gamma_0 (\delta_1 - \delta_0) \frac{\delta_1 - \delta_0}{1 + \delta_1},
\]

(28)

\[
\frac{\partial W'(y_0)}{\partial \gamma_1} = \gamma_1 (\delta_1 - \delta_0) \frac{\delta_1 - \delta_0}{1 + \delta_1}.
\]

(29)

Impact of the private benefits \( \phi_0 \):

Higher private benefits increase optimal precautionary liquidity: the derivative is given by

\[
\frac{\partial W'(y_0)}{\partial \phi_0} = 1.
\]

(30)
Proof of Lemma 1.. We solve for the equilibrium starting with $t=1$. We consider first the case where the liquidity supply constraint is binding ($I \leq aR$). The state where the liquidity shock does not arrive can again be ignored as the the banker then has no need for funds and will hence neither borrow nor liquidate. When the liquidity shock hits, a banker will generate a sufficient liquidity to just meet the liquidity requirement (equation (11)) is fulfilled with equality:

$$y_i + \delta_i + l_i = \lambda.$$  

(31)

In equilibrium, bankers have to be indifferent between generating liquidity through borrowing and through liquidations. The (net) cost of raising one unit of liquidity from investors is $(1 + \delta_i)(1 + \eta_i) - 1$, whereas the cost of raising it through liquidations is $(1 + \phi_i)\gamma(l)$. Setting both equal and solving for the date-1 interest rate we obtain:

$$\eta_i = \frac{\gamma(l) - \phi_i}{1 + \phi_i}.$$  

(32)

The interest rate is positive (follows from Assumption 1), reflecting that there is a shortage of liquidity supply. Investors will hence supply their entire date-1 liquidity, $I - (1 + \delta_i)y_i$, to banks. It follows that the total date-1 liquidity raised by banks, $y_i$, and total liquidations, $l_i$, are still characterized by equations (7) and (8).

We consider next $t=0$. Equilibrium requires that investors are indifferent between lending to banks and storing funds. From the equation for investors’ profits, (14), we obtain that this requires $r_0 = \phi_0$, that is, the excess return on liquidity at $t=0$ (earned for sure) has to equal the return at $t=1$ when the liquidity shortage materializes, times the likelihood of a liquidity shortage.

We next derive banker’s profit as a function of precautionary liquidity, $y_i^p$. Substituting $r_0$ and $\eta_i$ in (13) and using that $y_i = \lambda - y_0 - l_i$ (equation (31)) we obtain:

$$W^{B}(y_i^p) = R - ((1 + \delta_i)(1 + \pi\gamma(l) - \delta_i) - \phi_0)\gamma(l) - \pi(\lambda - y_0)\gamma(l).$$  

(33)

We consider next the case where the borrowing constraint is the binding one ($I > aR$). In the case of no liquidity shock at $t=1$, there are again no liquidity dealings between banks and investors as banks have no liquidity requirements. When the liquidity shock hits, banks will now be constrained by their borrowing capacity. Each bank will hence borrow up to its borrowing capacity, equation (12), and raise the remaining liquidity through liquidations. Since now the liquidity demand is constrained by the factor investor, investor liquidity is in excess supply. The equilibrium interest rate at $t=1$ hence has to be $\eta = 0$, in order to make investors indifferent between lending and storage.

Investors thus do not earn excess return on liquidity at $t=0$. This means that the interest rate at $t=0$ has to be zero as well (in order to make investors indifferent to lending at $t=0$ and $t=1$). The borrowing constraint hence simplifies to:

$$(1 + \delta_i)y_0 + (1 + \delta_i)y_i \leq \alpha R.$$  

(35)

Using that (35) will be fulfilled with equality at $t=1$ when the shock hits, one can see that total ex-post liquidity, $y_i$, and total liquidations, $l_i$, are still given by equation (7) and (12). Setting $\eta = 0$ in (31), we get for banker’s profits:

$$W^{B}(y_i^p) = R - (\delta_i - \phi_0)y_i - \pi\delta_iy_i - \pi\gamma(l)y_i.$$  

(36)

Differentiating (34) and (36) we obtain equation (15). □

Proof of Proposition 2.. When $\delta_i = \delta_0$, the social planner and a banker’s benefit from (precautionary) liquidity are identical ($W^*(y_i) = W^{B}(y_i)$). They face hence the same optimization problem and, therefore, we obtain efficiency of the equilibrium: $y_0^* = y_0^{BF}$. When $\delta_0 < \delta_i$, for all $y_0$ the social gains from liquidity exceed the banker’s one: $W^*(y_0) > W^{B}(y_0)$. Given that we have assumed interior solutions for the planner’s problem and given concavity of $W(y_0)$ and $W^{B}(y_0)$:

$$W^*(y_0) = -\pi[y_0^2\gamma'(l) + y_0\gamma''(l)] < 0,$$

(37)

$$W^{B}(y_0) = -\pi[y_0\gamma'(l)] < 0,$$

we thus have that $y_0^* > y_0^{BF}$. Similarly, for $\delta_0 > \delta_i$, we have that $W^*(y_0) < W^{B}(y_0)$ and hence $y_0^* < y_0^{BF}$. □

Proof of Proposition 3.. We note first that for the (constrained) efficient allocation, as analyzed in Section 3, it does not matter whether the stock of liquidity is public or private. The efficient allocation in an economy with investor liquidity $I$ and public liquidity $m$ is thus identical to the one of an economy with investor liquidity $I + m$. In the following we denote with $y_0^{BF}(m)$ the efficient ex-ante liquidity in an economy with liquidity $I + m$ (as determined in Section 3) and with $y^{ex}(m) (= \frac{I - \lambda}{1 + \delta_0})$ the corresponding ex-post liquidity. Similarly, we denote with $y_0^*(m)$ and $y^*(m)$ (Section 4) the equilibrium amounts of liquidity when total liquidity is $I + m$.

We assume that the public stock of liquidity is large enough to implement efficient liquidity at either $t=0$ or at $t=1$: $m \geq \max[y_0^{BF}(m), y^{ex}(m)]$.

We next analyze optimal CB facilities and whether they can implement the efficient solution. Three different cases have to be distinguished, depending on the stock of public liquidity and the relative stringency of the date-1 constraints.

Case 1: When all public liquidity $m$ is injected into the economy, no constraint binds (\( \lambda - \frac{\min[I, m]}{1 + \min[I, m]} \leq 0 \)).

This is the case where public liquidity is sufficiently large such that together with investor liquidity the liquidity demand $\lambda$ can be fulfilled and the borrowing capacity does not bind when $\lambda$ is raised. In this case, the CB can avoid liquidations entirely. We can determine the smallest $m$ that achieves this by rearranging $\lambda - \frac{\min[I, m]}{1 + \min[I, m]} \leq 0$ for $m$ (and using that the borrowing constraint does not bind):

$$\hat{m} = \lambda(1 + \min[\delta_0, \delta_1]) - I.$$  

(39)

The CB can then implement the efficient allocation by offering in total $\hat{m}$ at the same interest rate as the one that prevails in the market for private
liquidity. To see this, note that externalities across banks only arise because of liquidations (Section 4). In the absence of liquidations, the equilibrium amount of precautionary liquidity in an economy with liquidity \( I + m \) will hence be equal to the efficient one: \( y^e(\hat{m}) = y^e(\hat{m}) \). The CB can thus fully focus on inserting an amount of liquidity that is sufficient to resolve the liquidity shortage; it does not have to pursue an active interest rate policy in order to affect the allocation of liquidity.

Since – everything else equal – the CB prefers the policy with the lowest expected public liquidity provision, the CB will supply as much as possible of \( m \) ex-post (as ex-post liquidity has to be provided with probability \( p < 1 \) only). It will thus set up a facility of \( m_S^* = y^0(\hat{m}) \) at date 1, and will offer the rest at date 0: \( m_T^* = \hat{m} - y^0(\hat{m}) \). Since liquidations are absent, interest rates in the market for private liquidity will be zero. The CB can hence offer its liquidity at zero interest rates as well: \( r_{CB}^0 = r_{CB}^{eff} = 0 \).

**Case 2:** When all public liquidity \( m \) is injected into the economy, the liquidity supply constraint binds (\( I \leq \pi R \) and \( \lambda - \min[9, 11] > 0 \)).

In this case, liquidations cannot be avoided, and take place because of an insufficient supply of liquidity. In order to minimize liquidations as much as possible, the CB will then offer its entire stock: \( m^* = m \).

Can the CB implement the efficient allocation by lending at market rates – as in the first case? The answer is no. If the CB lends at market rates, the expected (private) return on liquidity is equalized across dates – as in the case without CB (condition (33) in the proof of Lemma 1). The private incentives to hold liquidity are then inefficient (for \( \delta_0 \neq \delta_i \)). Consider, alternatively, that the CB lends at below market rates; but that private lending still takes place at \( t = 0 \) and \( t = 1 \) (this will be the case when the borrowing facility is not too large). In this case, market borrowing will still be the marginal source of funding for banks (as banks will first take up the cheaper CB facility). The private return on liquidity will continue to be equalized across time and the incentives to hold precautionary liquidity remain inefficient.

What is the reason for the CB cannot improve the equilibrium outcomes in these two scenarios? Precautionary liquidity is in equilibrium determined by the relative ex-ante and ex-post cost of borrowing. As market borrowing is the marginal source of funding in both scenarios, private incentives pin down the attractiveness of precautionary liquidity. As a result, the CB cannot affect the allocation of liquidity. The only possibility for the CB to improve the market allocation is hence to fully replace private lending, either ex-ante or ex-post.

Consider first \( \delta_0 < \delta_i \). In this case precautionary liquidity is insufficient in a pure market equilibrium. The CB, however, can implement the efficient allocation by offering the efficient amount of liquidity ex-ante: \( m_S^* = y^0(\hat{m}) \). Since there is a liquidity shortage in the economy, market interest rates at date 1, and hence also at date 0, will be larger than zero. By setting sufficiently low interest rates on its ex-ante facility, the CB can make sure that banks will only borrow public liquidity. Consider in particular \( I_{CB}^0 = 0 \). Banks will then borrow the facility in full (since the allocation of liquidity across banks has no welfare consequences, the CB can ration banks in an arbitrary way). As there is a tendency for insufficient private liquidity, the market for ex-ante private liquidity will be inactive. Total precautionary liquidity at banks will hence be \( y^0(\hat{m}) \). At \( t = 1 \), the CB can then lend it remaining stock \( (m_T^* = \hat{m} - y^0(\hat{m})) \) at market interest rates. The market interest rate is the one that makes banks ex-post indifferent between borrowing and liquidation. 

Equation (32) tells us that this is the case when \( r_1^* = \frac{\gamma(1 - \delta_1)}{I + \delta_1} \). Given that liquidations are \( \hat{t} = \lambda - \frac{\gamma - I + k_0 y^0(\hat{m})}{I + \delta_1} \), the central bank can implement a market-neutral facility by setting \( r_{CB}^* = \frac{\gamma(1 - \delta_1)}{I + \delta_1} \) on its ex-post facility.

Consider next \( \delta_0 > \delta_i \). In this case precautionary liquidity is excessive in a pure market equilibrium, and there is insufficient ex-post liquidity. Consider that the CB opens a date-1 facility of \( m_S^* = y^0(\hat{m}) \) at zero interest rates \( (r_{CB} = 0) \) and lends its remaining stock \( (m_T^* = \hat{m} - y^0(\hat{m})) \) at market interest rates at \( t = 0 \). As there is a shortage of liquidity at date 1, banks will then take up the entire public supply at this date. Since private liquidity is undersupplied ex-post, no private lending will take place. The market interest rate at date 0 is then pinned down by the condition that banks have to be indifferent between raising liquidity ex-ante and liquidating ex-post. Similar to equation (32), this condition is \( r_0^* = \frac{\gamma(1 + \delta_0)}{I + \delta_1} \). The CB can thus offer its facility at a rate of \( r_0^* = \frac{\gamma(1 + \delta_0)}{I + \delta_1} \).

The reason for why the CB can implement efficiency is that, by lowering interest rates sufficiently, it can fully replace private liquidity at one of the two dates. Through setting the size of the lending facility, the CB can then control bank liquidity at one date, and given that overall liquidity is fixed, also at the other date.

**Case 3:** When all public liquidity \( m \) is injected into the economy, the borrowing constraint binds (\( I > \pi R \) and \( \lambda - \min[\pi R I, 1 + \min(\pi R I)] > 0 \)).

In this case, in order to minimize liquidations as much as possible, the CB will inject a total amount of liquidity that makes the borrowing constraint just bind: \( \hat{m} = \pi R + I \).

Consider first \( \delta_0 < \delta_i \). As equilibrium liquidity is too low, the CB would like to increase it by lending \( y^0(\hat{m}) \) at \( t = 0 \). However, as we have shown in Section 4, market interest rates are zero at both dates when the borrowing constraint binds. Thus the CB cannot lend at rates lower than the market. If it sets up a lending facility of \( y^0(\hat{m}) \) with \( r_{CB} = 0 \) at \( t = 0 \), banks will not take it up (in full). As the CB can then not affect the rates on marginal borrowings, the private returns on ex-ante and ex-post liquidity will remain equalized and banks will borrow precautionary liquidity \( y^0(\hat{m}) \) regardless of the characteristics of the borrowing facility. It follows that in this case the CB has no control over the allocation of liquidity and the economy will be characterized by insufficient precautionary liquidity. The CB can in this situation focus entirely on inserting the right amount of liquidity. To minimize the expected amount of liquidity provided, the CB will offer as much as possible ex-post. This implies setting \( m_S^* = y^0(\hat{m}) \) at \( t = 1 \), and offering the remaining stock \( m_T^* = \hat{m} - y^0(\hat{m}) \), at date 0. The interest rates at both dates are zero \( (r_{CB} = 0) \).

A similar argumentation applies to the case of \( \delta_0 > \delta_i \). The CB would now like to increase ex-post lending by replacing the market at \( t = 1 \). However, since interest rates are already at zero, this is not possible. We thus obtain the opposite efficiency result (excessive precautionary liquidity).

We can conclude that the efficiency bias in the case of the binding borrowing constraints is the same as in Section 4.21 Optimal liquidity policies are identical to the case of \( \delta_0 < \delta_i \); the central bank offers zero-interest rate facilities at date 0 and date 1 of \( m_S^* = \hat{m} - y^0(\hat{m}) \) and \( m_T^* = y^0(\hat{m}) \).

The proposition follows from combining the three cases.

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21 In order to implement the efficient allocation, the CB would need to have more control over the liquidity holdings at banks. For example, if the CB could fully control the market for private liquidity, it could implement optimal liquidity holdings by limiting the provision of private liquidity at either \( t = 0 \) or \( t = 1 \).
liquidity shortages. Outsiders value assets less than banks, which creates costs to liquidity shortages. Asset buyers may be non-bank financial institutions (vulture funds, private equity or hedge funds) or be located outside the financial system. The risk-bearing capacity of such buyers is likely to be limited – hence assets need to be increasingly discounted in order to be acquired.

Specifically, we consider a continuum of risk-averse outsiders of measure one (while we assume risk-aversion through preferences, risk aversion may also arise indirectly because of capital or leverage constraints). Outsiders are endowed with \( I_0 \) (\( > l \)) units of wealth at \( t = 1 \) and consume at \( t = 2 \). Their utility takes quadratic form:

\[
u_c = c_0 - \frac{1}{2} \rho c_1^2.
\]

with \( \rho > 0 \). We assume complete segmentation between outsiders and investors: outsiders can continue bank projects at \( t = 1 \) but do not lend to banks, while for investors it is vice versa. We also assume that outsiders are less efficient users of assets, in their hands the date-2 return falls by an amount \( \sigma > 0 \). In addition, assets become risky when held by outsiders: in case an outsider acquires assets from banks at \( t = 1 \), the date-2 return will be distributed with a common variance of 1.

We first consider the efficient outcome. We assume that the social planner cannot make outsiders worse off at \( t = 1 \) – similar to what is assumed for investors. In addition, the social planner cannot undertake redistributions among agents at \( t = 2 \) (otherwise he can eliminate any risks for outsiders through ex-post transfers with other agents). We denote with \( q \) the amount of assets transferred to outsiders at \( t = 1 \). Since each outsider’s valuation is decreasing in the amount of asset he is holding, it is optimal to distribute assets equally among outsiders. The total utility loss in the economy from outsiders taking on \( q \) units of assets is then \( q \gamma + \frac{1}{2} \rho q^2 \).

We assume that it is optimal to solve liquidity shortages using asset sales, rather than letting assets become worthless by not providing the liquidity injection (this will be the case for \( p \) and \( \varepsilon \) sufficiently small). We thus have for welfare:

\[
W(q_0, y_1, q) = R + I + I_0 - (\delta_0 - \phi_0) y_0 \sigma \delta_1 y_1 - \sigma (q + \frac{1}{2} \rho q^2).
\]

This equation is identical to welfare in the baseline model (equation (1)), except that total endowment is now \( I + I_0 \) and the liquidation cost has become \( q \gamma + \frac{1}{2} \rho q^2 \).

Given that an outsider’s utility from holding \( q \) units of assets is \( R q - q \gamma - \frac{1}{2} \rho q^2 \), his per-unit utility is \( R - \varepsilon - \frac{1}{2} \rho q \). Since outsiders cannot be made worse off, an asset can thus be transferred in return for an amount of date-1 liquidity of no more than \( R - \varepsilon - \frac{1}{2} \rho q \). In order to generate one unit of liquidity for banks, hence \( \frac{1}{\sigma} \) assets have to be transferred. Given the asset discount \( \varepsilon + \frac{1}{2} \rho \), the cost of generating one unit of liquidity is then given by \( \gamma(q) = \frac{\varepsilon + \frac{1}{2} \rho}{1 - \varepsilon - \frac{1}{2} \rho} \).

How many assets need to be transferred to investors at \( t = 1 \) to generate \( l \) units of liquidity? The maximum liquidity that can be raised when \( q \) units are transferred is \( R(q - \varepsilon - \frac{1}{2} \rho q) \). Hence to generate an amount of \( l \), one has to transfer assets such that the condition \( l = q(R - \varepsilon - \frac{1}{2} \rho q) \) is fulfilled. Solving for \( q \) gives \( q = \frac{1}{2} R - \varepsilon - \sqrt{(R - \varepsilon)^2 - 4pl} \). Using this to substitute \( q \) in \( \gamma(q) \) we obtain a new expression for liquidation costs:

\[
\gamma(l) = \frac{R + \varepsilon - \sqrt{(R - \varepsilon)^2 - 4pl}}{R - \varepsilon + \sqrt{(R - \varepsilon)^2 - 4pl}}.
\]

We thus have that (unit) liquidation costs increase in the total liquidity that is raised (\( \gamma(l) > 0 \)). We also have that \( \gamma(0) > 0 \) for sufficiently large \( \varepsilon \) and \( \gamma(l) > 0 \), hence this liquidation cost function can fulfill the assumption of the baseline model. Rewriting the loss from asset transfers in (41), \( \varepsilon + \frac{1}{2} \rho q^2 \), with \( \gamma(l) \) we obtain the same expression for welfare as in the baseline model (except that the endowment is now \( I + I_0 \)). The constraints to the optimization problem are unchanged; it follows that the same solution is obtained as in the baseline model.

Turning to the analysis of the equilibrium, we derive the outcome in the asset market (between banks and outsiders) at \( t = 1 \). This asset market is active when the liquidity shock arrives. Consider an equilibrium where \( q \) units of assets are transferred to outsiders at a price of \( p \). Given that outsider’s utility is symmetric and given that their utility is declining in the amount of the asset they are buying in the market, in an equilibrium all outsiders will acquire the same amount of assets. Since in equilibrium outsiders have to be indifferent to asset purchases, we hence have that the price equals an outsider’s (unit) valuation of the asset:

\[
p = R - \varepsilon - \frac{1}{2} \rho q.
\]

This is the same discount as for the social planner. The liquidation cost function is hence also the same as in the baseline model. It follows that the equilibrium outcome is unchanged.

Since both efficient solution and equilibrium are unchanged, Proposition 2 continues to hold. That is, equilibrium liquidity holdings are inefficient (unless \( \delta = \delta_1 \)). The source of the inefficiency is a standard fire-sale externality: an individual bank takes \( \gamma \) as given – but its liquidations (slightly) increase liquidation costs at all other banks.

References

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