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Generalizations of Schoenberg's theorem on positive definite kernels

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Abstract

The seminal theorem of I.J. Schoenberg characterizes positive definite (p.d.) kernels on the unit sphere S^{n-1} invariant under the automorphisms of the sphere. We obtain two generalizations of this theorem for p.d. kernels on fiber bundles. Our first theorem characterizes invariant p.d. kernels on bundles whose fiber is a product of a compact set and the unit sphere. This result implies, in particular, a characterization of invariant under the automorphisms of the sphere p.d. kernels on a product of S^{n-1} and a compact set. Our second result characterizes invariant p.d. kernels on the bundle whose fiber is S^{n-1} , base space is $(S^{n-1})^r$ and map is the projection on the base space. This set of kernels is isomorphic to the set of invariant under the automorphisms of the sphere continuous functions F on $(S^{n-1})^{r+2}$ such that $F(\cdot, \cdot, Z)$ is positive definite for every $Z \in (S^{n-1})^r$. When Z is fixed, this class reduces to the class of p.d. kernels invariant under the stabilizer of Z in the automorphism group of the sphere. For $r = 1$ these kernels have been used to obtain upper bounds for the spherical codes problem. Our extension for $r > 1$ can be used to construct new upper bounds on the size of spherical codes.

Keywords: Schoenberg's theorem; positive definite kernels, spherical codes

1 Introduction

The seminal theorem of I.J. Schoenberg [22] characterizes positive definite kernels on the unit sphere S^{n-1} invariant under the automorphisms of the sphere. A kernel on S^{n-1} is a continuous function $K : S^{n-1} \times S^{n-1} \rightarrow \mathbb{R}$ such that $F(x, y) = F(y, x)$ for all $x, y \in S^{n-1}$. A kernel is positive definite (p.d.) if for any finite $U \subset S^{n-1}$ the restriction of K to $U \times U$ is a

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positive semi-definite matrix. Let O_n be the orthogonal group in dimension n , which is the automorphism group of S^{n-1} . Schoenberg [22] proved that K is a positive definite kernel on S^{n-1} invariant under the action of O_n if and only if

$$K(x, y) = \sum_{i \in \mathbb{N}} c_i P_i^{\frac{n}{2}-1}(x^\top y), \quad c_i \geq 0,$$

where $P_i^{\frac{n}{2}-1}$ is a Gegenbauer polynomial of order $\frac{n}{2}-1$ and degree i .

In this paper we provide two generalizations of the above result to positive definite kernels on fiber bundles. Theorem 1 characterizes p.d. kernels on bundles whose fiber is a product of a compact set and the unit sphere. In particular, for any compact $V \subseteq \mathbb{R}^r$, we obtain a characterization of p.d. kernels on $V \times S^{n-1}$ which are invariant under the action of O_n when the arguments from V are fixed. Next, Theorem 2 characterizes the class of continuous functions $F(x, y, Z)$ on $(S^{n-1})^{r+2}$ invariant under O_n such that $F(\cdot, \cdot, Z)$ is a p.d. kernel for every $Z \in (S^{n-1})^r$. This class of functions can as well be viewed as kernels on a fiber bundle whose fiber is S^{n-1} , base space is $(S^{n-1})^r$ and map is the projection on the base space.

Our work is inspired by the connection of p.d. kernels to some combinatorial problems on infinite compact graphs. The well known linear programming upper bound for the kissing number problem by Delsarte et al. [10] can be obtained using Schoenberg's theorem 3. The extension of Schoenberg's theorem by Bachoc and Vallentin [3] is used to obtain the strongest existing semidefinite programming upper bounds on the kissing number [3, 16, 17]. The kissing number problem is a particular instance of the more general spherical codes problem. Schoenberg's theorem has been used to obtain bounds on spherical codes [2, 20], as well as bounds for other problems from coding theory and discrete geometry, such as binary codes [23], sphere packings [6, 7, 13], distance avoiding sets [9], measurable chromatic number [4], one-sided kissing number [19]. The extension of Schoenberg's theorem by Musin [18] has been used to obtain bounds for the maximum number of equiangular lines in \mathbb{R}^n [8].

Schoenberg's theorem has been generalized in several ways. First, let $r \geq 0$, and pick r distinct points in (S^{n-1}) . Consider p.d. kernels invariant under the automorphisms of the sphere fixing those points, that is the stabilizer of those points in O_n . Schoenberg's theorem describes the case $r = 0$, when no points are fixed. Next, Bachoc and Vallentin [3] characterized the case when K is a polynomial and $r = 1$ point is fixed. Finally, Musin [18] characterized the case $r \leq n - 2$. In this paper we extend this idea even further in Theorem 2. Namely, we consider the class of continuous functions $F(x, y, Z)$ on $(S^{n-1})^{r+2}$ such that $F(\cdot, \cdot, Z)$ is a p.d. kernel for every $Z \in (S^{n-1})^r$. There is a close connection between our result and [18], as for any fixed $Z \in (S^{n-1})^r$ we have that $F(\cdot, \cdot, Z)$ is a p.d. kernel invariant under the stabilizer of Z . Thus Musin's result characterizes $F(\cdot, \cdot, Z)$ for each fixed Z . However, it does not fully characterize F as the dependence on Z is not explicit in [18] since Z is assumed to be constant.

The approach in this paper differs from the approach by Musin [18]. Musin [18] uses modified Gegenbauer polynomials and the corresponding modification of the classical addition theorem for Gegenbauer polynomials [14]. On the contrary, we reduce the class of considered functions to the case where Schoenberg's theorem applies. To prove our results, we generalize the notion of a p.d. kernel from a kernel on a set to a kernel on a fiber bundle. Our results describe invariant under the action of O_n p.d. kernels on fiber bundles generated using the

unit sphere. This result can be also viewed as a continuation of another known extension of Schoenberg's theorem due to Bochner [5] who generalized the theorem for group invariant p.d. kernels on compact topological spaces.

The outline of the paper is as follows. In Section 2 we present the basic notation and concepts used throughout the paper and motivate our study. Section 3 introduces fiber bundles and kernels on them and presents our main Theorems 1 and 2. Section 4 contains major proofs. Further observations and ideas about future research are considered in Section (5).

2 Preliminaries and motivation

We denote the sets of real and nonnegative real numbers by \mathbb{R} and \mathbb{R}_+ , respectively. We denote the unit sphere in \mathbb{R}^n by S^{n-1} and equip it with the standard measure ω . Further, O_n is the orthogonal group in dimension n , and \mathbb{S}^n is the space of $n \times n$ symmetric matrices over \mathbb{R} .

A matrix $M \in \mathbb{S}^n$ is called positive semidefinite if $x^\top M x \geq 0$ for all $x \in \mathbb{R}^n$. We use the notation $M \succeq 0$ if M is positive semidefinite. We are interested in the infinite dimensional version of positive (semi-) definiteness. For any $V \subseteq \mathbb{R}^n$, let $C(V)$ be the set of real-valued continuous functions on V . We call kernels on V the set of symmetric real continuous functions on $V \times V$:

$$\mathcal{K}(V) = \{F \in C(V \times V) : F(x, y) = F(y, x), \forall x, y \in V\}.$$

Notice that for any finite set U of size n , the space $\mathcal{K}(U)$ of kernels on U is isomorphic to \mathbb{S}^n . A kernel K on V is positive definite (p.d.) if for any finite $U \subset V$ the restriction of K to $U \times U$ is positive semidefinite.

Proposition 1 (Lemma 1 in Bochner [5]). *Let $V \subset \mathbb{R}^n$ be a compact set equipped with a finite measure μ strictly positive on open subsets. Then K is a p.d. kernel on V if and only if for any $g(x) \in C(V)$,*

$$\int_V \int_V K(x, y) g(x) g(y) d\mu(x) d\mu(y) \geq 0.$$

Throughout the paper we use the following properties of p.d. kernels due to Schoenberg [21].

Proposition 2. *Let $V \subset \mathbb{R}^n$ be compact, then*

- (a) *A sum of finitely many p.d. kernels on V is a p.d. kernel.*
- (b) *A product of finitely many p.d. kernels on V is a p.d. kernel.*
- (c) *A continuous limit of a sequence of p.d. kernels on V is a p.d. kernel.*

Schoenberg [22] characterized p.d. kernels on the unit sphere invariant under the action of O_n in terms of Gegenbauer, or ultraspherical, polynomials. Gegenbauer polynomials $P_d^\alpha(t) : [-1, 1] \rightarrow \mathbb{R}$ of order α and degree d are inductively defined for any $\alpha \in \mathbb{R}$ and $d \geq 0$, as $P_0^\alpha(t) = 1, P_1^\alpha(t) = 2\alpha t$ and for $d > 1$

$$dP_d^\alpha(t) = 2t(d + \alpha - 1)P_{d-1}^\alpha(t) - (d + 2\alpha - 2)P_{d-2}^\alpha(t) \tag{1}$$

For each fixed order α , Gegenbauer polynomials form an orthogonal basis, with respect to the weight function $\nu = (1 - t^2)^{\alpha - \frac{1}{2}}$, for univariate polynomials on the interval $[-1, 1]$. Thus, every univariate polynomial $g(t)$ of degree d can be represented via its Gegenbauer polynomial expansion

$$g(t) = \sum_{i=0}^d c_i P_i^\alpha(t), \quad c_i \in \mathbb{R}. \quad (2)$$

We denote the standard L^2 norm of P_i^α by $p_{\alpha,i} := \int_{-1}^1 |P_k^{\frac{n}{2}-1}(t)|^2 (1 - t^2)^{\alpha - \frac{1}{2}} dt$.

Proposition 3 (Schoenberg [22]). *Let $n \geq 2$. The kernel $K \in \mathcal{K}(S^{n-1})$ is invariant under the action of O^n and p.d. if and only if there exists $c_i \geq 0$ for $i = 0, 1, \dots$ such that*

$$K(x, y) = \sum_{i \in \mathbb{N}} c_i P_i^{\frac{n}{2}-1}(x^\top y), \quad (3)$$

where the series converges absolutely uniformly. Also, the coefficients of the expansion 3 are given by

$$c_k = \frac{1}{p_{n/2-1,k}} \int_{x,y \in S^{n-1}} K(x, y) P_k^{\frac{n}{2}-1}(x^\top y) d\omega(x) d\omega(y). \quad (4)$$

Given $Z = [z_1, \dots, z_r] \in (S^{n-1})^r$, let

$$\text{Stab}_{O_n}(Z) = \{M \in O_n : MZ = Z\}$$

be the stabilizer of Z in O_n . Musin [18] extended Schoenberg's theorem by characterizing p.d. kernels on the unit sphere invariant under the action of $\text{Stab}_{O_n}(Z)$. To present this extension theorem, denote by $\mathcal{R}(Z)$ the range of Z , let $\Pi_Z = Z(Z^\top Z)^{-1}Z^\top$ be the orthogonal projection onto $\mathcal{R}(Z)$, and let $\Pi_Z^\perp = I - \Pi_Z$, where I is the identity matrix, be the orthogonal projection onto $\mathcal{R}(Z)^\perp$.

Proposition 4 (Musin [18]). *Let $n \geq 2$, and let $Z \in (S^{n-2})^r$ be of rank r . The kernel $K \in \mathcal{K}(S^{n-1})$ is invariant under the action of $\text{Stab}_{O_n}(Z)$ and p.d. if and only if there exist p.d. kernels c_i on $\{x \in \mathbb{R}^r : z = Z^\top y, y \in S^{n-1}\}$ for $i = 0, 1, \dots$ such that*

$$K(x, y) = \sum_{i \in \mathbb{N}} c_i(Z^\top x, Z^\top y) P_i^{\frac{n-r}{2}-1} \left(\frac{(\Pi_Z^\perp x)^\top \Pi_Z^\perp y}{\|\Pi_Z^\perp x\| \|\Pi_Z^\perp y\|} \right). \quad (5)$$

When $r = 1$ and K is a polynomial, the result in Proposition 4 follows from the decomposition by Bachoc and Vallentin [3], who use classical results on spherical harmonics, see, e.g., [1, Chapter 9].

Propositions 3 and 4 were used to obtain new upper bounds on the spherical codes problem. In this problem, the number $A(n, \theta)$ of points on S^{n-1} is maximized, for which the pairwise angular distance is not smaller than some value θ . Schoenberg's theorem (Proposition 3) leads to the linear programming upper bound for the spherical codes problem by Delsarte

et al. [10], and Musin’s theorem (Proposition 4) when $r = 1$ leads to the semi-definite programming bounds by Bachoc and Vallentin [3]. Our findings in this paper are motivated by similar upper bounds on the spherical codes problem by Kuryatnikova and Vera [15] based on the copositive reformulation of the problem by Dobre et al. [11]. The bound in [15] exploits the so called 2-p.d. functions; For any $r \geq 0$ and $V \subseteq \mathbb{R}^n$, a function $F \in C(V^{r+2})$ is called 2-p.d. on V if for all $Z \in V^r$, $F(\cdot, \cdot, Z)$ is a p.d. kernel on V .

Let $V = S^{n-1}$, $r \geq 1$, and let F be a 2-p.d. function on the unit sphere invariant under the action of O_n . Implementing the upper bound in [15] requires a characterization of F . It is clear that for every fixed $Z \in (S^{n-1})^r$, $F(\cdot, \cdot, Z)$ is invariant under $\text{Stab}_{O_n}(Z)$, and thus has the form as in Proposition 4. However, in the context of [15] Z is variable. The question is then how to modify Proposition 4 to make explicit the dependance on Z .

Although this fact is not stated explicitly, in Proposition 4 functions c_i can differ for different choices of Z . More precisely, for each orbit $Z^{O_n} = \{MZ : M \in O_n\}$ of $Z \in S^{n-1}$ we have a different $c_i^{Z^{O_n}}$, but since Z is fixed, this dependence of c_i on the orbit of Z is implicit in Proposition 4. We generalize Proposition 4 taking this dependence into account in order to characterize 2-p.d. functions. In order to simplify our presentation, we consider a generalization of p.d. kernels on a set to p.d. kernels on a fiber bundle (see next Section 3 for precise definitions). We characterize p.d. kernels on fiber bundles generated by products of the unit sphere and compact sets. We use this result to characterize the set of 2-p.d. functions on the unit sphere invariant under the action of O_n .

3 Kernels on bundles

We think on kernels parameterized by a set of parameters B . Here not only the kernel depends on the parameters from B , but also the domain of the kernel might depend on the choice of the parameters from B .

3.1 Bundles and their properties

A bundle is a map $f : A \rightarrow B$, where A is called the total space and B is called the base space [12]. For each $b \in B$, $A_b := f^{-1}(b) \subset A$ is called the fiber over b . We think of A_b as representing a set “parameterized by” b . Our definition of a bundle is quite unrestrictive. In particular, we do not ask the fibers to be homeomorphic. If for every $b \in B$ the set A_b is compact, we say that the bundle $f : A \rightarrow B$ is compact. In this paper we restrict ourselves to the case of compact bundles, where $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$.

Definition 1. *As examples, we define the following bundles which we frequently use in the sequel.*

1. Given $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, let $\pi_{A,B} : A \times B \rightarrow B$ be the projection bundle defined by $\pi_{A,B}(a, b) := b$.
2. Given bundles $f_1 : A_1 \rightarrow B$ and $f_2 : A_2 \rightarrow B$, define

$$A_1 \otimes A_2 := \{(a_1, a_2) \in A_1 \times A_2 : f_1(a_1) = f_2(a_2)\}.$$

We define the fiber product bundle $f_1 \otimes f_2 : A_1 \otimes A_2 \rightarrow B$ as

$$f_1 \otimes f_2(a_1, a_2) := f_1(a_1).$$

The fiber product is also called the Whitney sum. It has the property that for every $b \in B$, $(A_1 \otimes A_2)_b = (A_1)_b \times (A_2)_b$.

3. Given a bundle $f : A \rightarrow B$ and $U \in \mathbb{R}^k$, let $U \times f : U \times A \rightarrow B$ be the bundle such that $U \times f(u, a) := f(a)$. In the case $U = S^{n-1}$, we call $S^{n-1} \times f$ a cylinder.

Now, we introduce the notion of kernel on a bundle. The idea is that for each $b \in B$ we have a kernel on A_b , and the dependence on b is continuous. Given a bundle $f : A \rightarrow B$, we define a kernel on f to be a continuous map $K : A \otimes A \rightarrow \mathbb{R}$ such that $K_b : A_b \times A_b \rightarrow \mathbb{R}$ is a kernel for each $b \in B$. We say K is p.d. on f if K_b is p.d. for each $b \in B$.

We say that a bundle $f : U \rightarrow B$ is a subbundle of $g : A \rightarrow B$ if $U \subseteq A$, and $f = g|_U$. We call $f : U \rightarrow B$ a projection subbundle if it is a subbundle of some projection bundle.

Remark 1. Given any projection bundle $\pi_{A,B} : A \times B \rightarrow B$, we have $\pi_{A,B} \otimes \pi_{A,B} \cong \pi_{A \times A, B}$, and thus every p.d. kernel on a subbundle $f : U \rightarrow B$ of $\pi_{A,B}$ is in correspondence with a continuous map $K : U \times U \times B \rightarrow \mathbb{R}$ such that for each $b \in B$, $K(\cdot, \cdot, b) = K_b$ is a p.d. kernel on U_b . In the sequel we abuse the notation and make no difference between a kernel on projection subbundle $f : U \rightarrow B$ and its corresponding continuous map.

Our last definition is the action of a group on a bundle. Given bundle $f : A \rightarrow B$ and group G , for G to act on f means that G acts both on A and on B , and both actions are consistent with f . That is, for all $g \in G$ and $a \in A$, $f(a)^g = f(a^g)$. We denote the orbit of $a \in A$ under G by $a^G := \{a^g : g \in G\}$, and let $\mathcal{O}_G(A) = \{a^G : a \in A\}$ be the set of orbits of A . We define b^G and $\mathcal{O}_G(B)$ analogously.

When G acts on a bundle $f : A \rightarrow B$, it is natural to define the G -orbit bundle of f as $\mathcal{O}_G(f) : \mathcal{O}_G(A) \rightarrow \mathcal{O}_G(B)$ such that $\mathcal{O}_G(f)(a^G) = f(a)^G$. Notice that in the G -orbit bundle, for any $b^G \in \mathcal{O}_G(B)$ we have $[\mathcal{O}_G(A)]_{b^G} = \mathcal{O}_G(A_b)$.

Now, we propose an extension of the group acting on B to a group acting on projection bundle $\pi_{A,B}$. Assume G acts on B . We define the vertical action of G on $\pi_{A,B}$ by fixing the elements of A ; that is, for all $a \in A$, $b \in B$ and $g \in G$ we define the action of G on $A \times B$ by $(a, b)^g := (a, b^g)$. Notice that this action and the action of G on B are consistent with $\pi_{A,B}$, and thus define an action on $\pi_{A,B}$. Moreover, $\mathcal{O}_G(\pi_{A,B}) = \pi_{A, \mathcal{O}_G(B)}$. For any projection subbundle $f : U \rightarrow B$, we say that G acts vertically on f if the action of G is the restriction of the vertical action on the corresponding projection bundle. Notice that this is the case only if for any $b \in B$ and $g \in G$ we have $U_b = U_{b^g}$.

In general, looking at $\mathcal{O}_G(f)$ is not enough to characterize p.d. kernels on f invariant under the action of G as kernels are bivariate functions and thus one should look at 2-orbits, instead of 1-orbits. One exception is the case of vertical actions, as the following straightforward proposition shows.

Proposition 5. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$. Assume that G is a topological group that acts on B , and endows $\mathcal{O}_G(B)$ with the usual topology. Let $f : A \rightarrow B$ be a projection subbundle

such that G acts vertically on f . Let K be a kernel on f invariant under the vertical action of G . Define the function K^G as

$$K^G(a_1, a_2, b^G) := K(a_1, a_2, b) \text{ for all } b \in B, a_1, a_2 \in A_b.$$

Then K^G is a kernel on $\mathcal{O}_G(f)$, and K is p.d. if and only if K^G is p.d.

Proof. The result follows from the definition of the vertical action and the G -orbit bundle. \square

This work is motivated by 2-p.d. functions introduced in [15]. Remark 1 explains that 2-p.d. functions on A are p.d. kernels on π_{A, A^r} for some given r . The interest in [15] is in $(r+2)$ -variate 2-p.d. functions on S^{n-1} invariant under the natural action of O_n on $(S^{n-1})^{r+2}$. In the language of kernels on fiber bundles, those are p.d. kernels on $\pi_{S^{n-1}, (S^{n-1})^r}$ invariant under the natural action of O_n on $\pi_{S^{n-1}, (S^{n-1})^r}$. In the rest of this paper we characterize such kernels.

3.2 Main results

Next we present two generalizations of Schoenberg's theorem (Proposition 3). Given a bundle $f : A \rightarrow B$, we define the horizontal action of O_n on the cylinder ${}_{S^{n-1}}\times f$ by $(x, a)^M = (Mx, a)$ and $b^M = b$, for each $x \in S^{n-1}$, $a \in A$, $b \in B$ and $M \in O_n$. In our first Theorem we characterize the p.d. kernels on cylinders, invariant under the horizontal action of O_n .

Theorem 1. *Let $n \geq 2$, $r > 0$, $m > 0$, and let $A \in \mathbb{R}^r, B \in \mathbb{R}^m$. Let $f : A \rightarrow B$ be a compact bundle. Then a kernel K on the cylinder ${}_{S^{n-1}}\times f$ is p.d. and invariant under the horizontal action of O_n if and only if there are p.d. kernels c_i on f , $i = 0, 1, \dots$ such that for all $u_1, u_2 \in S^{n-1}$, all $b \in B$ and $a_1, a_2 \in A_b$*

$$K_b([\begin{smallmatrix} a_1 \\ u_1 \end{smallmatrix}], [\begin{smallmatrix} a_2 \\ u_2 \end{smallmatrix}]) = \sum_{i \in \mathbb{N}} (c_i)_b(a_1, a_2) P_i^{\frac{n}{2}-1}(u_1^\top u_2). \quad (6)$$

Also, the coefficients of the expansion are given by

$$(c_k)_b(a_1, a_2) = \frac{1}{p_{n/2-1, k}} \int_{u_1, u_2 \in S^{n-1}} K_b([\begin{smallmatrix} a_1 \\ u_1 \end{smallmatrix}], [\begin{smallmatrix} a_2 \\ u_2 \end{smallmatrix}]) P_k^{\frac{n}{2}-1}(u_1^\top u_2) d\omega(u_1) d\omega(u_2). \quad (7)$$

Notice that $\pi_{S^{n-1}, (S^{n-1})^r}$ is isomorphic to the cylinder ${}_{S^{n-1}}\times \text{id} (S^{n-1})^r$, where $\text{id} (S^{n-1})^r$ is the identity bundle on $(S^{n-1})^r$. Consider the (natural) action of O_n on $\pi_{S^{n-1}, (S^{n-1})^r}$. This action is not horizontal, and thus Theorem 1 does not apply. Our second theorem describes p.d. kernels on the bundle $\pi_{S^{n-1}, (S^{n-1})^r}$ which are invariant under the action of O_n . Given $r > 0$, define

$$\Lambda^r = \{Y \in \mathbb{S}^r : Y \succeq 0, Y_{ii} = 1 \text{ for all } i \in \{1, \dots, r\}\}.$$

Theorem 2. *Let $r \geq 0$ and $n \geq r + 2$. A kernel K on $\pi_{S^{n-1}, (S^{n-1})^r}$ is invariant under the action of O_n and p.d. if and only if there are p.d. kernels c_i , $i = 0, 1, \dots$ on the projection subbundle $f : \left\{ \left[\begin{smallmatrix} 1 & y^\top \\ y & Y \end{smallmatrix} \right] \in \Lambda^{r+1} : Y \succ 0 \right\} \rightarrow \{Y \in \Lambda^r : Y \succ 0\}$, such that for all $x, y \in S^{n-1}$ and $Z \in (S^{n-1})^r$,*

$$K_Z(x, y) = \sum_{i \in \mathbb{N}} (c_i)_{Z^\top Z} (Z^\top x, Z^\top y) P_i^{\frac{n-r}{2}-1} \left(\frac{(\Pi_Z^\perp x)^\top \Pi_Z^\perp y}{\|\Pi_Z^\perp x\| \|\Pi_Z^\perp y\|} \right). \quad (8)$$

The expressions in Theorem 2 and Proposition 4 are very similar. Indeed, the difference is that Proposition 4 does not present c_i as a function of $Z^\top Z$ since Z is assumed to be fixed. Notice that Proposition 4 follows from Theorem 2, as given any $Z \in (S^{n-2})^r$ and any p.d. kernel K on S^{n-1} invariant under the action of $\text{Stab}_{O_n}(Z)$, there is p.d. kernel \hat{K} on $\pi_{S^{n-1}, (S^{n-1})^r}$ invariant under the action of O_n such that $\hat{K}_Z(x, y) = K(x, y)$.

4 Proofs of main theorems

In this section we present the proofs of Theorems 1 and 2.

4.1 Proof of Theorem 1

Let K be a kernel on $S^{n-1} \times f$. The “only if” part of the statement follows from Propositions 2 and 3. To prove the converse, let K be p.d. and invariant under the horizontal action of O_n . Let $b \in B$ and $a_1, a_2 \in A_b$. The kernel

$$G_b^{a_1, a_2}(u_1, u_2) = K_b\left(\begin{bmatrix} a_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ u_2 \end{bmatrix}\right) + K_b\left(\begin{bmatrix} a_2 \\ u_1 \end{bmatrix}, \begin{bmatrix} a_1 \\ u_2 \end{bmatrix}\right) + K_b\left(\begin{bmatrix} a_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} a_1 \\ u_2 \end{bmatrix}\right) + K_b\left(\begin{bmatrix} a_2 \\ u_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ u_2 \end{bmatrix}\right)$$

is p.d. on S^{n-1} and invariant under O_n . From Schoenberg’s theorem (Proposition 3) we have

$$G_b^{a_1, a_2}(u_1, u_2) = \sum_{k \geq 0} (d_i)_b(a_1, a_2) P_k^{\frac{n}{2}-1}(u_1^\top u_2), \quad (9)$$

where the $(d_i)_b(a_1, a_2)$ are nonnegative, and the series (9) converges absolutely uniformly.

As O_n acts transitively on S^{n-1} , we have

$$K_b\left(\begin{bmatrix} a_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ u_2 \end{bmatrix}\right) = K_b\left(\begin{bmatrix} a_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} a_2 \\ u_1 \end{bmatrix}\right) = K_b\left(\begin{bmatrix} a_2 \\ u_1 \end{bmatrix}, \begin{bmatrix} a_1 \\ u_2 \end{bmatrix}\right),$$

thus,

$$\begin{aligned} G_b^{a_1, a_2}(u_1, u_2) &= 2K_b\left(\begin{bmatrix} a_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ u_2 \end{bmatrix}\right) + K_b\left(\begin{bmatrix} a_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} a_1 \\ u_2 \end{bmatrix}\right) + K_b\left(\begin{bmatrix} a_2 \\ u_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ u_2 \end{bmatrix}\right) \\ G_b^{a_i, a_i}(u_1, u_2) &= 4K_b\left(\begin{bmatrix} a_i \\ u_1 \end{bmatrix}, \begin{bmatrix} a_i \\ u_2 \end{bmatrix}\right) \end{aligned} \quad (i = 1, 2).$$

Defining $(c_k)_b(a_1, a_2) = (d_k)_b(a_1, a_2) - \frac{1}{4}(d_k)_b(a_1, a_1) - \frac{1}{4}(d_k)_b(a_2, a_2)$ and using (9), we obtain

$$\begin{aligned} K_b\left(\begin{bmatrix} a_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ u_2 \end{bmatrix}\right) &= \frac{1}{2} \left(G_b^{a_1, a_2}(u_1, u_2) - \frac{1}{4} G_b^{a_1, a_1}(u_1, u_2) - \frac{1}{4} G_b^{a_2, a_2}(u_1, u_2) \right) \\ &= \frac{1}{2} \sum_{k \in \mathbb{N}} (c_k)_b(a_1, a_2) P_k^{\frac{n}{2}-1}(u_1^\top u_2). \end{aligned} \quad (10)$$

Remark 2. Notice that Schoenberg’s theorem can not be applied directly to $K_b\left(\begin{bmatrix} a_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ u_2 \end{bmatrix}\right)$, as this is not necessarily a p.d. kernel for all $b \in B$ and $a_1, a_2 \in A_b$. Intuitively the reason for this is that this function does not correspond to a “principal submatrix” of K_b when $a_1 \neq a_2$.

Next, we argue that c_k ’s are p.d. kernels on f . Fix $k \geq 0$, then we claim that

Claim 1. For every $b \in B$, $a_1, a_2 \in A_b$,

$$(c_k)_b(a_1, a_2) = \frac{1}{P_{n/2-1, k}} \int_{u_1, u_2 \in S^{n-1}} K_b\left(\begin{bmatrix} a_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ u_2 \end{bmatrix}\right) P_k^{\frac{n}{2}-1}(u_1^\top u_2) d\omega(u_1) d\omega(u_2)$$

Fix $b \in B$. Claim 1 and the continuity of K imply that $(c_k)_b$ is continuous. From (7) it follows that $(c_k)_b(a_1, a_2) = (c_k)_b(a_2, a_1)$ for all $a_1, a_2 \in A_b$. Hence $(c_k)_b$ is a kernel.

From our assumptions, A_b is compact. We use Proposition 1 to show that $(c_k)_b$ is p.d. on $S^{n-1} \times A_b$. Let $h \in C(A_b)$ be given, then

$$\begin{aligned} & \int_{a_1, a_2 \in A_b} (c_k)_b(a_1, a_2) h(a_1) h(a_2) d\mu(a_1) d\mu(a_2) \\ &= \int_{\substack{a_1, a_2 \in A_b \\ u_1, u_2 \in S^{n-1}}} K_b\left(\begin{bmatrix} a_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ u_2 \end{bmatrix}\right) P_k^{\frac{n}{2}-1}(u_1^\top u_2) h(a_1) h(a_2) d\omega(u_1) \dots d\mu(a_2) \\ &\geq 0. \end{aligned}$$

The inequality holds since K_b and $P_k^{\frac{n}{2}-1}$ are p.d. on $S^{n-1} \times A_b$, and from Proposition 2.b their product is p.d. too.

To finish, we prove Claim 1. We know that $\{P_i^{\frac{n}{2}-1}\}_{i \in \mathbb{N}}$ is a sequence of orthogonal polynomials on S^{n-1} under ω . Fix $k \geq 0$, $b \in B$ and $a_1, a_2 \in A_b$. From Schoenberg's theorem 3 and the definition of $(c_i)_b(a_1, a_2)$ in 10, we have that $\sum_{i \geq 0} (c_i)_b(a_1, a_2) P_i^{\frac{n}{2}-1}(u_1^\top u_2)$ converges absolutely uniformly on $S^{n-1} \times S^{n-1}$. Hence, as P_k is continuous on $[-1, 1]$ and therefore bounded, the series

$$\sum_{i \geq 0} (c_i)_b(a_1, a_2) P_i^{\frac{n}{2}-1}(u_1^\top u_2) P_k^{\frac{n}{2}-1}(u_1^\top u_2)$$

converges absolutely uniformly too. Therefore

$$\begin{aligned} & \int_{u_1, u_2 \in S^{n-1}} K_b\left(\begin{bmatrix} a_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ u_2 \end{bmatrix}, b\right) P_k^{\frac{n}{2}-1}(u_1^\top u_2) d\omega(u_1) d\omega(u_2) \\ &= \int_{u_1, u_2 \in S^{n-1}} \sum_{i \geq 0} (c_i)_b(a_1, a_2) P_i^{\frac{n}{2}-1}(u_1^\top u_2) P_k^{\frac{n}{2}-1}(u_1^\top u_2) d\omega(u_1) d\omega(u_2) \\ &= \sum_{i \geq 0} (c_i)_b(a_1, a_2) \int_{u_1, u_2 \in S^{n-1}} P_i^{\frac{n}{2}-1}(u_1^\top u_2) P_k^{\frac{n}{2}-1}(u_1^\top u_2) d\omega(u_1) d\omega(u_2) \\ &= p_{n/2-1, k}(c_k)_b(a_1, a_2). \end{aligned}$$

4.2 Proof of Theorem 2

The idea of the proof is to relate kernels on $\pi_{S^{n-1}, (S^{n-1})^r}$ to kernels on cylinders over S^{n-r-1} and apply Theorem 1. We do this via continuous transformations using Lemmas 1 and 2 below.

Lemma 1. *Let $f : C \rightarrow B$ be a compact bundle. Let $T : A \rightarrow C$ be a continuous function. Let K be a kernel on f . Define L by $L_b(a_1, a_2) = K_b(T(a_1), T(a_2))$ for all $b \in B$, $a_1, a_2 \in A_b$.*

1. L is a kernel on $f \circ T$.
2. If K is p.d. on f , then L is p.d. on $f \circ T$.

3. If T is surjective and L is p.d. on $f \circ T$, then K is p.d. on f .

Proof. As T and K are continuous, L is continuous by definition. Also, for any $b \in B$ and any $a_1, a_2 \in A_b$, we have $L_b(a_1, a_2) = K_b(T(a_1), T(a_2)) = K_b(T(a_2), T(a_1)) = L_b(a_2, a_1)$. Thus L is a kernel on $f \circ T$. Now assume K is p.d. on f . For any $k > 0$, any $b \in B$, and any $a_1, a_2, \dots, a_k \in A_b$, we have that $[L_b(a_i, a_j)]_{1 \leq i, j \leq k} = [K_b(T(a_i), T(a_j))]_{1 \leq i, j \leq k} \succeq 0$. Thus L is p.d. on $f \circ T$. Now assume T is surjective and L is p.d. on $f \circ T$. Then given $k > 0$, any $b \in B$, and any $c_1, c_2, \dots, c_k \in C_b$ there are $a_1, a_2, \dots, a_k \in A_b$ such that $c_i = T(a_i)$. Thus the matrix $[K_b(c_i, c_j)]_{1 \leq i, j \leq k} = [L_b(a_i, a_j)]_{1 \leq i, j \leq k}$ is p.d.. Therefore K is p.d. on f . \square

Given $Z \in (S^{n-1})^r$. For any $x \in \mathbb{R}^n$ we can write $x = \Pi_Z x + \Pi_Z^\perp x$, where $\Pi_Z x \in \mathcal{R}(Z)$ and $\Pi_Z^\perp x \in \mathcal{R}(Z)^\perp$. To prove Theorem 2, we exploit the fact that $\text{Stab}_{O_n}(Z)$ fixes $\Pi_Z S^{n-1}$ and acts transitively on $\Pi_Z^\perp S^{n-1}$. Therefore in the following steps, for every $x \in S^{n-1}$ we separate the fixed component $\Pi_Z x$ and exploit the symmetry of the varying component $\Pi_Z^\perp x$.

We have $\dim(\mathcal{R}(Z)^\perp) = n - \dim(\mathcal{R}(Z)) = n - \text{rank } Z$, and therefore $\mathcal{R}(Z)^\perp$ is isomorphic to $\mathbb{R}^{n-\text{rank } Z}$. Namely, there is an isomorphism ϕ_Z between $\mathbb{R}^{n-\text{rank } Z-1}$ and $\mathcal{R}(Z)^\perp$. Analogously, $\mathcal{R}(Z)$ has dimension $\text{rank } Z$ and thus is isomorphic to $\mathbb{R}^{\text{rank } Z}$, and there is an isomorphism γ_Z between $\mathbb{R}^{\text{rank } Z}$ and $\mathcal{R}(Z)$. Let

$$\mathcal{S} = \{Z \in (S^{n-1})^r : \text{rank } Z = r\}.$$

The set \mathcal{S} is dense in $(S^{n-1})^r$. In the sequel we restrict most of our arguments to \mathcal{S} , to avoid ‘‘singularities’’ in further proofs and definitions. In particular, the dependence on Z of the isomorphisms ϕ_Z and γ_Z can not be continuous in the whole $(S^{n-1})^r$, but when we restrict ourselves to \mathcal{S} , ϕ_Z and γ_Z can be chosen continuous. For instance, let $Ort : \mathbb{R}^{n \times r} \rightarrow \mathbb{R}^{n \times (n-r)}$ be such that $Ort(Z)$ provides an orthonormal basis of $\mathcal{R}(Z)^\perp$ for any $Z \in \mathbb{R}^{n \times r}$ of rank r . Then the isomorphism between $\mathcal{R}(Z)^\perp$ and \mathbb{R}^{n-r} can be viewed as the bijection:

$$\phi_Z : \mathbb{R}^{n-r} \rightarrow \mathcal{R}(Z)^\perp, \quad \phi_Z(v) = Ort(Z)v \text{ for all } v \in \mathbb{R}^{n-r}.$$

We can construct a continuous isomorphism between \mathbb{R}^r and $\mathcal{R}(Z)$ as the following bijection

$$\gamma_Z : \mathbb{R}^r \rightarrow \mathcal{R}(Z), \quad \gamma_Z(u) = Z(Z^\top Z)^{-1}u \text{ for all } u \in \mathbb{R}^r.$$

We are particularly interested in the isomorphism between $\Pi_Z S^{n-1} \subset \mathcal{R}(Z)$ and

$$B_Z := \gamma_Z^{-1} \circ \Pi_Z S^{n-1} = \{Z^\top x : x \in S^{n-1}\}.$$

Notice that for any $Z \in \mathcal{S}$ we can send $x \in \Pi_Z S^{n-1}$ to the unit sphere in \mathbb{R}^{n-r} by normalizing x . Then, since $\text{Stab}_{O_n}(Z)$ is isomorphic to O_{n-r} , any action of $\text{Stab}_{O_n}(Z)$ on $\Pi_Z^\perp S^{n-1}$ can be associated with an action of O_{n-r} on S^{n-r-1} . Hence we can use the result for the orthogonal group acting on the unit sphere, described in Theorem 1. To formalize this procedure, we need the following lemma.

Lemma 2. *The maps*

$$T_1 : S^{n-r-1} \times \{(u, Z) : Z \in \mathcal{S}, u \in B_Z\} \rightarrow \{(x, Z) : Z \in \mathcal{S}, x \in S^{n-1}\}$$

$$\begin{bmatrix} v \\ Z \end{bmatrix} \mapsto \begin{bmatrix} \phi_Z(v) \sqrt{1 - \|\gamma_Z(u)\|^2} \\ Z \end{bmatrix}$$

$$T_2 : \{(x, Z) : Z \in \mathcal{S}, x \in S^{n-1} \setminus \mathcal{R}(Z)\} \rightarrow S^{n-r-1} \times \{(u, Z) : Z \in \mathcal{S}, u \in B_Z\}$$

$$\begin{bmatrix} x \\ Z \end{bmatrix} \mapsto \begin{bmatrix} \frac{\phi_Z^{-1}(\Pi_Z^\perp x)}{\|\phi_Z^{-1}(\Pi_Z^\perp x)\|} \\ \gamma_Z^{-1}(\Pi_Z x) \\ Z \end{bmatrix}$$

are continuous, T_1 is surjective, T_2 is injective, and $T_1 \circ T_2 = \text{id}_{\{(x,Z):Z \in \mathcal{S}, x \in S^{n-1} \setminus \mathcal{R}(Z)\}}$.

Proof. Continuity follows from the continuity of γ_Z , ϕ_Z and their inverses. It is a straightforward calculation to check that T_1 is a surjection and $T_1 \circ T_2 = \text{id}_{\{(x,Z):Z \in \mathcal{S}, x \in S^{n-1} \setminus \mathcal{R}(Z)\}}$. T_2 is injective since for any $x_1, x_2 \in \mathbb{R}^n$, if $x_1 \neq x_2$ then either $\Pi_Z x_1 \neq \Pi_Z x_2$ or $\Pi_Z^\perp x_1 \neq \Pi_Z^\perp x_2$. \square

Proof of Theorem 2. First, denote

$$\mathcal{B} := \{(u, Z) : Z \in \mathcal{S}, u \in B_Z\} = \{(Z^\top x, Z) : Z \in \mathcal{S}, x \in S^{n-1}\},$$

and consider the compact projection subbundle $f : \mathcal{B} \rightarrow \mathcal{S}$. Let K be a kernel on $\pi_{S^{n-1}, (S^{n-1})^r}$. Throughout the proof we use a function L on the cylinder $_{S^{n-r-1}} \times f$ defined by

$$L_Z \left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right) = K \left(T_1 \left(\begin{bmatrix} u_1 \\ v_1 \\ Z \end{bmatrix} \right), T_1 \left(\begin{bmatrix} u_2 \\ v_2 \\ Z \end{bmatrix} \right) \right), \quad (11)$$

for all $v_1, v_2 \in S^{n-r-1}$, $Z \in \mathcal{S}$, $u_1, u_2 \in B_Z$. By our assumptions K is a kernel on $\pi_{S^{n-1}, (S^{n-1})^r}$. Since T_1 is surjective by Lemma 2, we have that L is a kernel on $_{S^{n-r-1}} \times f$ by Lemma 1.

For the ‘‘only if’’ direction of the theorem, let K be a kernel on $\pi_{S^{n-1}, (S^{n-1})^r}$ that has expansion (8). Then K is invariant under the action of O_n as it is continuous and depends only on inner products of x, y, z_1, \dots, z_r on a dense subset of its domain. To show that K is p.d., notice that expansion (8) of K implies L has an expansion of type (6). Hence L is a p.d. kernel on $_{S^{n-r-1}} \times f$ by Theorem 1. Therefore K is p.d. when restricted to a kernel on $\pi_{S^{n-1}, \mathcal{S}}$ by Lemma 1, and thus K is p.d. by continuity.

For the ‘‘if’’ direction of the theorem, let K be p.d. on the projection bundle $\pi_{S^{n-1}, (S^{n-1})^r}$ and invariant under the action of O_n . That is, $K_{MZ}(Mx, My) = K_Z(x, y)$ for all $M \in O_n, x, y \in S^{n-1}$ and $Z \in (S^{n-1})^r$. Then for any $Z \in (S^{n-1})^r$, K_Z is invariant under the action of $\text{Stab}_{O_n}(Z)$. Using Lemma 1, we obtain that L is a p.d. kernel on $_{S^{n-r-1}} \times f$. As $\text{Stab}_{O_n}(Z)$ fixes $\Pi_Z S^{n-1}$ and acts transitively on $S^{n-1} \cap \mathcal{R}(Z)^\perp$, we have that L is invariant under the horizontal action of O_{n-r} on $_{S^{n-r-1}} \times f$. From Theorem 1 there are p.d. kernels d_i on f , $i = 0, 1, \dots$ such that for all $v_1, v_2 \in S^{n-r-1}$, all $Z \in \mathcal{S}$ and $u_1, u_2 \in B_Z$

$$L_Z \left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right) = \sum_{i \geq 0} (d_i)_Z(u_1, u_2) P_i^{\frac{n-r}{2}-1}(v_1^\top v_2). \quad (12)$$

Now, $T_1 \circ T_2 = \text{id}_{\{(x,Z):Z \in \mathcal{S}, x \in S^{n-1} \setminus \mathcal{R}(Z)\}}$ from Lemma 2. Thus for any $Z \in \mathcal{S}$ and $x_1, x_2 \in S^{n-1} \setminus \mathcal{R}(Z)$, we have

$$\begin{aligned} K_Z(x_1, x_2) &= L_Z \left(T_2 \left(\begin{bmatrix} x_1 \\ Z \end{bmatrix} \right), T_2 \left(\begin{bmatrix} x_2 \\ Z \end{bmatrix} \right) \right) \\ &= \sum_{i \geq 0} (d_i)_Z \left(\gamma_Z^{-1}(\Pi_Z x_1), \gamma_Z^{-1}(\Pi_Z x_2) \right) P_i^{\frac{n-r}{2}-1} \left(\left(\frac{\phi_Z^{-1}(\Pi_Z^\perp x_1)}{\|\Pi_Z^\perp x_1\|} \right)^\top \frac{\phi_Z^{-1}(\Pi_Z^\perp x_2)}{\|\Pi_Z^\perp x_2\|} \right) \\ &= \sum_{i \geq 0} (d_i)_Z (Z^\top x_1, Z^\top x_2) P_i^{\frac{n-r}{2}-1} \left(\frac{(\Pi_Z^\perp x_1)^\top \Pi_Z^\perp x_2}{\|\Pi_Z^\perp x_1\| \|\Pi_Z^\perp x_2\|} \right), \end{aligned} \quad (13)$$

where we have used $\phi_Z^{-1}(a_1)^\top \phi_Z^{-1}(a_2) = a_1^\top a_2$.

To finish, we specify the form of the d_i 's. Let $k \geq 0$. By Lemma 1 we have that d_k is a p.d. kernel on $f : \mathcal{B} \rightarrow \mathcal{S}$. Notice that O_n acts vertically on f since for every $M \in O_n$, $x \in S^{n-1}$ and $Z \in \mathcal{S}$, we have $(Z^\top x, Z)^M = (Z^\top x, MZ)$. Next we show that d_k is invariant under this vertical action. That is, that for all $Z \in \mathcal{S}$, $v_1, v_2 \in B_Z$ and $M \in O_n$ we have $(d_k)_Z(v_1, v_2) = (d_k)_{MZ}(v_1, v_2)$. By (7) from Theorem 1,

$$(d_k)_{MZ}(v_1, v_2) = \frac{1}{P_{(n-r)/2-1, k}} \int_{u_1, u_2 \in S^{n-r-1}} L_{MZ} \left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right) P_k^{\frac{n-r}{2}-1} (v_1^\top v_2) d\omega(v_1) d\omega(v_2).$$

Therefore it is enough to show that L is invariant under the vertical action of O_n on $S^{n-r-1} \times f$. By construction of ϕ_Z, γ_Z we have

$$\phi_{MZ}(v) = M\phi_Z(v), \quad \gamma_{MZ}(u) = M\gamma_Z(u), \quad \text{and} \quad \|\gamma_{MZ}(u)\| = \|\gamma_Z(u)\|.$$

Therefore

$$\begin{aligned} L_{MZ} \left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right) &= K \left(T_1 \left(\begin{bmatrix} u_1 \\ v_1 \\ MZ \end{bmatrix} \right), T_1 \left(\begin{bmatrix} u_2 \\ v_2 \\ MZ \end{bmatrix} \right) \right) \\ &= K \left(\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} T_1 \left(\begin{bmatrix} u_1 \\ v_1 \\ Z \end{bmatrix} \right), \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} T_1 \left(\begin{bmatrix} u_2 \\ v_2 \\ Z \end{bmatrix} \right) \right) = L_Z \left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right), \end{aligned}$$

where the last but one equality holds by invariance of K under O_n . Now, we can reduce ourselves to the orbit bundle of f under the vertical action of O_n . We have that $\mathcal{O}_{O_n}(\mathcal{S}) = \text{int } \Lambda^r$, $\mathcal{O}_{O_n}(\mathcal{B}) = \left\{ \begin{bmatrix} 1 & y^\top \\ y & Y \end{bmatrix} \in \Lambda^{r+1} : Y \succ 0 \right\}$ and $\mathcal{O}_{O_n}(f)$ is a projection. From Proposition 5 there is a p.d. kernel c_i on $\mathcal{O}_{O_n}(f) : \left\{ \begin{bmatrix} 1 & y^\top \\ y & Y \end{bmatrix} \in \Lambda^{r+1} : Y \succ 0 \right\} \rightarrow \{Y \in \Lambda^r : Y \succ 0\}$ such that for all $Z \in \mathcal{S}$, $v_1, v_2 \in B_Z$ we have that $(c_k)_{Z^\top Z}(v_1, v_2) = (d_k)_Z(v_1, v_2)$. \square

5 Further observations

One question for further research is what shape the coefficients c_i from expansion (8) could have. We show one result in this direction: a Gegenbauer polynomial of order $\left(\frac{n-r}{2}-1\right)$ used in (8) on $\pi_{S^{n-1}, (S^{n-1})^r}$ can be considered as a p.d. kernel on $\pi_{S^{n-1}, (S^{n-1})^{r+1}}$, and therefore it can be expressed in a series of the form (8) with coefficients c_i of a particular but rather complex form. To simplify the notation, given $x, y \in S^{n-1}$ and $Z \in (S^{n-1})^r$, define

$$\langle x, y \rangle_Z := (\Pi_Z^\perp x)^\top \Pi_Z^\perp y = x^\top y - (Z^\top x)^\top (Z^\top Z)^{-1} Z^\top y.$$

Proposition 6. *Let $r \in \mathbb{N}$, $x, y, q \in S^{n-1}$ and $Z \in (S^{n-1})^r$, then*

$$\begin{aligned} P_k^{\frac{n-r}{2}-1} \left(\frac{\langle x, y \rangle_Z}{\sqrt{\langle x, x \rangle_Z} \sqrt{\langle y, y \rangle_Z}} \right) &= \sum_{i=0}^k c_{k,i}^{n,r} \left(\frac{\langle x, x \rangle_{[Zq]}}{\langle x, x \rangle_Z} \right)^{i/2} \left(\frac{\langle y, y \rangle_{[Zq]}}{\langle y, y \rangle_Z} \right)^{i/2} P_i^{\frac{n-r}{2}-\frac{3}{2}} \left(\frac{\langle x, y \rangle_{[Zq]}}{\sqrt{\langle x, x \rangle_{[Zq]}} \sqrt{\langle y, y \rangle_{[Zq]}}} \right) \\ &\quad \cdot P_{k-i}^{\frac{n-r}{2}+i-1} \left(\frac{\langle x, q \rangle_Z}{\sqrt{\langle x, x \rangle_Z} \sqrt{\langle q, q \rangle_Z}} \right) P_{k-i}^{\frac{n-r}{2}+i-1} \left(\frac{\langle y, q \rangle_Z}{\sqrt{\langle y, y \rangle_Z} \sqrt{\langle q, q \rangle_Z}} \right) \end{aligned}$$

where $c_{k,i}^{n,r}$ are positive constants.

Proof. We use the addition theorem for Gegenbauer polynomials. Let $\alpha > 0$ and $k \in \mathbb{N}$, then for angles γ, θ, τ

$$\begin{aligned} & P_k^\alpha(\cos \theta \cos \tau + \sin \theta \sin \tau \cos \gamma) \\ &= \sum_{i=0}^k c_{k,i}^\alpha (\sin \theta)^i (\sin \tau)^i P_i^{\alpha-\frac{1}{2}}(\cos \gamma) P_{k-i}^{\alpha+i}(\cos \theta) P_{k-i}^{\alpha+i}(\cos \tau), \end{aligned}$$

where $c_{k,i}^\alpha$ are some positive constants that depend on α, k, i . More details about the formula can be found in [14]. Define

$$\begin{aligned} \cos \gamma &= \frac{\langle x, y \rangle_{[Zq]}}{\sqrt{\langle x, x \rangle_{[Zq]}} \sqrt{\langle y, y \rangle_{[Zq]}}} \\ \cos \theta &= \frac{\langle x, q \rangle_Z}{\sqrt{\langle x, x \rangle_Z} \sqrt{\langle q, q \rangle_Z}}, \quad \sin \theta = \sqrt{1 - \cos^2 \theta}, \\ \cos \tau &= \frac{\langle y, q \rangle_Z}{\sqrt{\langle y, y \rangle_Z} \sqrt{\langle q, q \rangle_Z}}, \quad \sin \tau = \sqrt{1 - \cos^2 \tau}. \end{aligned}$$

Now, expanding the expressions for the sines above and using the inverse formula for block matrices, one can show that

$$\sin \theta = \sqrt{\frac{\langle x, x \rangle_{[Zq]}}{\langle x, x \rangle_Z}}$$

and

$$\cos \theta \cos \tau + \sin \theta \sin \tau \cos \gamma = \frac{\langle x, y \rangle_Z}{\sqrt{\langle x, x \rangle_Z} \sqrt{\langle y, y \rangle_Z}}.$$

Substituting the corresponding expressions and orders of Gegenbauer polynomials in the addition formula with $\alpha = \frac{n-r}{2} - 1$, the result follows. \square

The proof of Proposition 6 is based on the addition theorem for Gegenbauer polynomials [14] and is related to the approach in Musin [18]. Namely, Musin [18] modifies the addition theorem for Gegenbauer polynomials to characterize p.d. kernels on S^{n-1} invariant under the action of $\text{Stab}_{O_n}(Z)$ for a given $Z \in (S^{n-1})^r$.

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