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# Concave/convex weighting and utility functions for risk: A new light on classical theorems

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ABSTRACT

This paper analyzes concave and convex utility and probability distortion functions for decision under risk (law-invariant functionals). We characterize concave utility for virtually all existing models, and concave/convex probability distortion functions for rank-dependent utility and prospect theory in complete generality, through an appealing and well-known condition (convexity of preference, i.e., quasiconcavity of the functional). Unlike preceding results, we do not need to presuppose any continuity, let be differentiability.

An example of a new light shed on classical results: whereas, in general, convexity/concavity with respect to probability mixing is mathematically distinct from convexity/concavity with respect to outcome mixing, in Yaari's dual theory (i.e., Wang's premium principle) these conditions are not only dual, as was well-known, but also logically equivalent, which had not been known before.



## 1. Introduction

This paper provides a tool for analyzing convexity and concavity of probability distortion and utility for decision under risk and for law-invariant functionals. We generalize all existing characterizations of concavity/convexity and make them more appealing. Unlike preceding results, we do not need to presuppose any continuity, let be differentiability. Those mathematical conditions are known to be problematic for empirical preference axiomatizations and empirical tests (§3). Continuity is nevertheless commonly imposed in the literature to simplify the mathematics. In our results, it is optional. The main tool in our analysis is Lemma 3, an adaptation of Theorem 3 of Wakker and Yang (2019) from uncertainty to risk.

Our Theorem 7 concerns Quiggin's (1982) rank-dependent utility and Tversky and Kahneman's (1992) prospect theory. It shows that the probability distortion function is concave/convex if and only if we have convexity/concavity of preference with respect to probabilistic mixing. It is remarkable that these well-known conditions are logically equivalent in full generality. Preceding results always assumed continuity.

Theorem 4 characterizes concave utility for Miyamoto's (1988) biseparable utility for risk. An attractive property of Miyamoto's model is that it comprises many existing models that are all special cases of it (Wakker, 2010 Observation 7.11.1). We thus characterize concave utility for all these models: original prospect theory (Kahneman and Tversky, 1979) for gains and for losses, Quiggin's (1982) rank-dependent utility, and prospective reference theory (Viscusi, 1989), Tversky and Kahneman's (1992) prospect theory for risk for gains and losses, disappointment aversion (Gul, 1992), Luce's (2000) binary RDU, RAM and TAX models (Birnbaum, 2008), and reference-dependent preferences (Köszegi and Rabin, 2006) for 0-kinked universal gain-loss functions. Also included is disappointment theory (Bell, 1985; Loomes and Sugden, 1986; for a disappointment function kinked at 0). Our characterizing preference condition does not need probabilities as inputs. Hence, it can be used for law-invariant functionals (i.e., Machina and Schmeidler, 1992 probabilistic sophistication), where probabilities can be subjective and, therefore, not directly observable, as for instance in Boonen and Ghossoub (2021).

Our Theorem 5 adapts one of the most appealing results in the literature on nonadditive measures, by Chateauneuf and Tallon (2002), to our context of risk, and an appealing result follows again: the common properties of convexity of probability distortion and concavity of utility are jointly characterized by the well-known convexity with respect to outcome mixing. We again bring in our generalizations of not needing continuity or differentiability, leav-

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0167-6687/

ing those optional. We similarly generalize related results by Chew and Mao (1995).

There is a well-known relation between risk theory and welfare theory. We can reinterpret probabilities as parts of a population at a particular wealth level. This way, theorems from risk can be transferred to welfare and vice versa. Risk aversion is reinterpreted as inequality aversion. The well-known Gini index of inequality in welfare was a precursor of Quiggin's rank-dependent utility. Ebert (2004) independently obtained results close to Chateauneuf and Tallon (2002) and Chew and Mao (1995) for welfare, and we similarly generalize his results.

There also is a tight relation between risk attitudes and Artzner et al.'s (1999) risk measures (Belles-Sampera et al., 2016; Goovaerts et al., 2010a). Thus, our results can be applied to risk measures, including the convex class introduced by Liu et al. (2020). Section 5 shows that, whereas properties of risk measures have commonly been derived from outcome mixtures or additions, probability mixtures provide an alternative tool. This gives, for instance, a new way to analyze Wang's premium principle (Cheung et al., 2020). For applications to reinsurance-design problems, see Liu et al. (2020 §4). A big pro is that our alternative approach does not need continua of outcomes, but can handle discrete outcome sets. Denuit et al. (2019 §2.2) emphasize the importance of such outcome sets for insurance.

Wakker and Yang (2019) analyzed convexity and concavity for the context of uncertainty, rather than risk, in a way similar to this paper. This paper will be self-contained and can be read independently.[1]

## 2. Basic definitions

We consider decision under risk with a set $\mathcal{P}$ of probability distributions, called *lotteries* (generic notation $P, Q, R$), over a set $X$ of *outcomes* (generic notation $\alpha, \beta, \gamma$, or $x_j$).[2] $X$ can be finite or infinite, and its elements can be monetary or non-monetary. We assume that $\mathcal{P}$ contains all *simple* probability distributions, assigning probability 1 to a finite subset of $X$, with generic notation $(p_1 : x_1, \ldots, p_n : x_n)$, and possibly more distributions. Measure-theoretic structure, with a ($\sigma$-)algebra on $X$, and lotteries only defined thereon, can be added at will, changing nothing in the analysis of this paper. Our analysis will focus on simple lotteries, where measure theoretic aspects are trivial. For non-simple lotteries, our analysis does not impose restrictions so that, again, measure theoretic conditions can be added at will.

A *preference relation*, i.e., a binary relation $\succcurlyeq$ on $\mathcal{P}$, is given; $\succ, \preccurlyeq, \prec, \sim$ are as usual. $V$ *represents* $\succcurlyeq$ if $V : \mathcal{P} \to \mathbb{R}$ satisfies $P \succcurlyeq Q \Leftrightarrow V(P) \geq V(Q)$ for all lotteries $P, Q \in \mathcal{P}$. This implies weak ordering on $\mathcal{P}$; i.e., $\succcurlyeq$ is transitive and complete. Outcomes $\alpha$ are identified with degenerate lotteries $(1 : \alpha)$. Thus, $\succcurlyeq$ also denotes preferences over outcomes.

A *(probability) distortion function* $w$ maps $[0, 1]$ to $[0, 1]$, is strictly increasing, and satisfies $w(0) = 0$ and $w(1) = 1$. We do not assume continuity of $w$. Discontinuities at $p = 0$ and $p = 1$ are of special empirical interest. For a distortion function $w$, and a function $U : X \to \mathbb{R}$, the *rank-dependent utility* (*RDU*) of a lottery $P$ is

$$\int_{\mathbb{R}^+} w(P(U(\alpha) > \mu))d\mu - \int_{\mathbb{R}^-} (1 - w(P(U(\alpha) > \mu)))d\mu. \quad (1)$$

For a simple lottery $(p_1 : x_1, \ldots, p_n : x_n)$ with $U(x_1) \geq \cdots \geq U(x_n)$, the RDU can be rewritten as

$$\sum_{j=1}^{n} (w(p_1 + \cdots + p_j) - w(p_1 + \cdots + p_{j-1}))U(x_j). \quad (2)$$

*Rank-dependent utility* (*RDU*) holds if there exist $w$ and $U$ such that $RDU$ represents $\succcurlyeq$. Then $U$ is the *utility function*, and it represents $\succcurlyeq$ on $X$. The special case of RDU with $w$ the identity is called *expected utility* (*EU*). We impose one more restriction on $\mathcal{P}$: RDU is well defined and finite for all its elements. A necessary and sufficient condition directly in terms of preferences—requiring preference continuity with respect to truncations of lotteries—is in Wakker (1993). A sufficient condition is that all lotteries are bounded (with an upper and lower bound contained in $X$).

**Assumption 1** *(Structural assumption).* $\mathcal{P}$ is a set of lotteries over outcome set $X$ containing all simple probability distributions. RDU holds. $X$ contains at least three nonindifferent outcomes $\gamma \succ \beta \succ \alpha$. □

## 3. Outcome mixing

This section considers outcome mixing for risk. For this purpose, we reinforce our assumptions.

**Assumption 2** *(Structural assumption for monetary outcomes).* Assumption 1 holds. Further, $X = I$ is a nonpoint interval and $U$ is strictly increasing. □

We do not presuppose continuity of $U$. Unlike virtually all axiomatizations in the literature we, similarly, do not need to assume continuity of the preference relation (except in Corollary 6). Many authors have warned against the problematic empirical status of continuity assumptions in preference axiomatizations (Ghirardato and Marinacci, 2001a; Halpern, 1999; Khan and Uyanik, 2021; Krantz et al., 1971 §9.1; Pfanzagl, 1968 §6.6 and §9.5; Wakker, 1988). The assumption is not merely technical but adds empirical content to the empirical axioms, and the problem is that it is unknown what that added empirical content is. Hence, given the purpose of preference axiomatizations to reveal the empirical content of theories, it is desirable to do without continuity if possible.[3] In our case, continuity of $U$ on $int(I)$ comes free of charge, following from the empirical axioms. At extremes ($min(I)$ for concavity and $max(I)$ for convexity) we have it optional. If continuity is considered to be desirable there, then we can get it by adding the corresponding continuity condition for $\succcurlyeq$. For simplicity, we restrict the definition of convex preferences to simple lotteries, which will be strong enough to give all the desired implications. Preceding papers (e.g., Yaari, 1987) defined the condition for risk by specifying an underlying state space and then extended the condition to nonsimple lotteries.[4] As we will show, imposing the

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[1] We thank an anonymous editor for recommending this approach.

[2] Lotteries may result from *law-invariant nonadditive measures* $W(.)$, i.e., $W = w(P(.))$ for a probability measure $P$ and a strictly increasing transformation $w$. This way, our analysis includes an important subclass of uncertainty models, which are probabilistically sophisticated in the sense of Machina and Schmeidler (1992).

[3] We do assume RDU, or biseparable utility, in our results. Köbberling and Wakker (2003) provided preference axiomatizations that do not assume continuity, but a weaker solvability condition. This condition still has observability problems, but to a lesser extent than continuity.

[4] For completeness, we give details. In the following results, the proofs of sufficiency of the preference conditions then remain unaltered because we only need simple lotteries for those. For necessity, concavity of $U$ and convexity of $w$ imply that the representing functional is concave (as in the proof of Theorem 5 or Wakker and Yang, 2019 Lemma B.1) and, hence, surely quasiconcave. Then $\succcurlyeq$ is convex. This implies, in particular, that convexity for all simple lotteries is equivalent to convexity for all lotteries under RDU. For further extensions to nonsimple lotteries, see Mao and Hu (2012). Alternative outcome-operations without a state space can be obtained by taking probabilistically independent combinations of lotteries (Goovaerts et al., 2010b).

preference conditions only on simple lotteries is enough to give all desired results. We can thus avoid the complications of defining underlying state spaces.

Because it is common in decision under risk to let concavity and convexity refer to probabilistic mixing, considered in the next section, we use a different term for outcome mixing. We call $\succcurlyeq$ *outcome-convex* if for each probability vector $p_1, \ldots, p_n$ (assumed to add to 1) and $0 < \lambda < 1$ we have

$$(p_1 : x_1, \ldots, p_n : x_n) \succcurlyeq (p_1 : y_1, \ldots, p_n : y_n) \Rightarrow$$
$$(p_1 : \lambda x_1 + (1-\lambda)y_1, \ldots, p_n : \lambda x_n + (1-\lambda)y_n \succcurlyeq (p_1 : y_1, \ldots, p_n : y_n). \quad (3)$$

We call $\succcurlyeq$ *outcome-convex on* $\mathcal{P}' \subset \mathcal{P}$ if Eq. (3) holds whenever all lotteries in it are contained in $\mathcal{P}'$. We next show that this extension of convexity in Euclidean domains to the lottery domain gives convenient axiomatizations of widely used properties. A new result on Yaari's (1987) analog of this extension is given in the next section (Corollary 8).

A *comoncone* is a subset of lotteries $\{(p_1 : x_1, \ldots, p_n : x_n) : x_1 \geq \cdots \geq x_n\}$, with $n \geq 2$ fixed, the probability vector $p_1, \ldots, p_n$ fixed, and $0 < p_1 < 1$. We call $\succcurlyeq$ *comonotonic outcome-convex* if it is outcome-convex on every comoncone; that is, if Eq. (3) holds whenever $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$. The following lemma provides a tool used throughout this paper.

**Lemma 3.** *Consider a comoncone* $\{(p_1 : x_1, \ldots, p_n : x_n) : x_1 \geq \cdots \geq x_n\}$. *Under Assumption 2, U is concave if and only if* $\succcurlyeq$ *is outcome-convex on this comoncone.* □

Ghirardato and Marinacci (2001b) propagated the biseparable utility model for uncertainty. For risk, this was done by Miyamoto (1988), who used the term generic utility. He also emphasized that any result for his theory holds for the many models comprised, referenced in our introduction. We now apply our technique to his model. For a fixed $0 \leq p \leq 1$, we denote by $\mathcal{P}_p$ the set of binary lotteries $\gamma_p\beta = (p : \gamma, 1-p : \beta)$, and by $\mathcal{P}_p^{\uparrow}$ we denote the subset with $\gamma \geq \beta$—it is a comoncone if $0 < p < 1$. *Biseparable utility* holds if there exist a utility function $U$ and a distortion function $w$ such that $RDU(\gamma_p\beta) = w(p)U(\gamma) + (1 - w(p))U(\beta)$ (for $\gamma \succcurlyeq \beta$) represents $\succcurlyeq$ on the set of all binary lotteries.

**Theorem 4.** *If Structural Assumption 2 holds except that biseparable utility holds instead of RDU, then U is concave if and only if* $\succcurlyeq$ *is outcome-convex on every* $\mathcal{P}_p^{\uparrow}$. *This holds if and only if* $\succcurlyeq$ *is outcome-convex on one set* $\mathcal{P}_p^{\uparrow}$ *with* $0 < p < 1$. □

Thus, we have characterized concave utility for virtually all existing models of risky choice (see introduction). For RDU, the preference conditions are equivalent to the stronger comonotonic outcome-convexity, as is easily verified. For EU, it is equivalent to the even stronger outcome-convexity. For EU, this result amounts to an alternative to the traditional characterizations based on weak risk aversion (preference for expected value) or strong risk aversion (aversion to mean-preserving spreads).

The preceding theorem characterized concavity of utility for RDU (and other theories), and Theorem 7 will characterize convexity of probability distortion for RDU. The following theorem efficiently characterizes the two properties jointly. Such "pessimistic" functionals have been widely used to represent downside risks, rather than overall preference values (Goovaerts et al., 2010a).

**Theorem 5.** *Under Assumption 2, U is concave and w is convex if and only if* $\succcurlyeq$ *is outcome-convex.* □

Theorem 5 captures the two most-studied properties of RDU through one basic preference condition. The result applies in particular to the widely studied law-invariant coherent risk measures that are comonotonically additive (Cornilly et al., 2018). The theorem shows that quasiconvexity of a functional, which is equivalent to convexity of the preference relation, has surprisingly strong implications. We need not assume continuity because it is implied (except at some extremes). This way, many results in the literature can be generalized, e.g. in Bellini et al. (2021) and Hu and Chen (2021 Remark 2.2.) In particular, it is a law-invariant convex risk functional in the sense of Liu et al. (2020). Chateauneuf and Tallon (2002) analyzed in detail how these properties underly various forms of diversification, important for risk functionals (Liu et al., 2020 §2.3). Ettlin et al. (2020) analyzed the role of diversification in optimal risk-sharing across networks of insurance companies.

Theorem 5 provides an interesting alternative to Chew, Karni, & Safra (1987). They showed, assuming differentiability, that concavity of $U$ plus convexity of $w$ is equivalent to aversion to mean-preserving spreads. Quiggin (1993 §6.2) provided an alternative proof, also assuming differentiability. Schmidt and Zank (2008) provided yet another proof, the only one available in the literature that did not assume differentiability; they still did assume continuity. It is desirable to avoid differentiability assumptions in preference axiomatizations because differentiability is even more problematic than continuity: unlike with continuity, for differentiability there is not even a preference condition to axiomatize it. Our derivations, therefore, neither assume differentiability. We obtain the following corollary, where for the definition of continuity and aversion to mean-preserving spreads we refer to Chew et al. (1987). We need to assume continuity because without it there are no results available in the literature on aversion to mean-preserving spreads.

**Corollary 6.** *Under Assumption 2 and continuity, outcome-convexity of* $\succcurlyeq$ *is equivalent to aversion to mean-preserving spreads.*

The result is remarkable because, at first sight, one condition concerns only outcome mixing whereas the other condition also involves probabilistic mixing. This surprising point was discussed by Quiggin (1993 §9.2) in a somewhat different context. Many papers have used aversion to mean-preserving spreads conditions in various forms. At the end of the appendix, we give details of several results relevant to the preceding analyses. Our paper shows that convexity conditions can serve as appealing alternative. This holds especially if the probabilities involved in mean-preserving spreads are subjective, implying that they are not directly observable, contrary to our preference condition. Thus, our condition can, for instance, serve to make the conditions in §5 of Gul and Pesendorfer (2015) directly observable. Gul and Pesendorfer (2015) used subjective probabilities as inputs in their axioms but subjective probabilities are not directly observable.

## 4. Probabilistic mixing

This section considers probabilistic mixing. For lotteries $P, Q$, $\lambda P \oplus (1-\lambda)Q$ denotes the probability measure assigning probability $\lambda P(\alpha) + (1-\lambda)Q(\alpha)$ to each outcome $\alpha$, with a similar probability mix for each subset of outcomes instead of $\{\alpha\}$. Probabilistic mixing can be defined for general outcome sets $X$.

We call $\succcurlyeq$ *convex* if $P \succcurlyeq Q \Rightarrow \lambda P \oplus (1-\lambda)Q \succcurlyeq Q$. This property, suggesting a deliberate preference for randomization, has been widely studied in the literature (Agranov and Ortoleva, 2017; Cerreia-Vioglio et al., 2019; Fudenberg et al., 2015; Machina, 1985; Saito, 2015; Sopher and Narramore, 2000). The opposite condition is more commonly found empirically: $\succcurlyeq$ is *concave* if $P \succcurlyeq Q \Rightarrow$

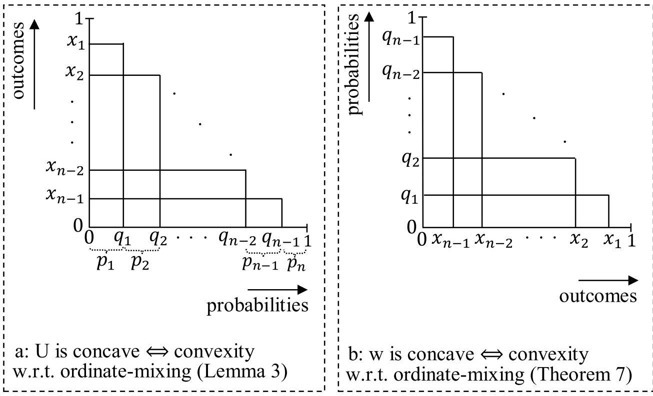


**Fig. 1.** Duality of probabilities versus outcomes, and $w$ versus $U$, for outcome interval $I=[0,1]$ and $U(0)=0$, $U(1)=1$. RDU results from Fig. (a) by transforming the abscissa by $w$ and the ordinate by $U$, and then calculating the area of the figure. RDU results from Fig. (b) by transforming the abscissa by $U$ and the ordinate by $w$, and then calculating the area of the figure.

$P \succcurlyeq \lambda P \oplus (1-\lambda)Q$. It is widely studied for distorted risk measures (Tsanakas, 2008).

To prepare for the following theorem, we explain a remarkable duality between outcomes and probabilities (Fig. 1). We focus on a compact outcome interval that we may normalize: $I=[0,1]$, with utility also normalized: $U(0)=0$ and $U(1)=1$. We can always assume the minimal outcome 0 to be present in the lottery below (Eq. (4)), i.e., $x_n=0$, by setting $p_n=0$ if necessary. We write $x_0=1$. Further, $q_j=p_1+\cdots+p_j$. It is called the *rank* of outcome $x_{j+1}$, being the probability of receiving an outcome ranked better. We write $q_0=0$. The RDU value of the lottery

$$(p_1:x_1,\ldots,p_n:x_n), \tag{4}$$

with utility function $U$ and probability transformation $w$ is

$$\sum_{j=1}^{n-1}(w(q_j)-w(q_{j-1}))U(x_j) \tag{5}$$

(Fig. 1a). By rearranging terms, it is

$$\sum_{j=1}^{n-1}(U(x_{n-j})-U(x_{n-j+1}))w(q_{n-j}). \tag{6}$$

But this is exactly the RDU value of the lottery

$$((x_{n-1}-x_n):q_{n-1},(x_{n-2}-x_{n-1}):q_{n-2},\ldots,(x_0-x_1):q_0) \tag{7}$$

with utility function $w$ and probability distortion function $U$ (Fig. 1b). Now the $x_j$s play the role of rank, with their differences outcome probabilities, and the $q_{n-j}$s play the role of outcome. Outcomes and ranks, i.e., the $x_j$s and $q_{n-j}$s, play a dual role. Outcome mixing is dual to probabilistic mixing. Using this duality, every theorem about probability distortion gives a theorem about utility, and vice versa.

To illustrate the above duality, assume Structural Assumption 2. Then we have convexity of $\succcurlyeq$ for lotteries in Eq. (4) if and only if for lotteries in Eq. (7) we have outcome-convexity. By Lemma 3, the latter holds if and only if the "utility function" $w$ in Eq. (6) is concave. We have shown that convexity of $\succcurlyeq$ is equivalent to concavity of probability distortion. However, we did so under Assumption 2. We can obtain the result in complete generality, under Assumption 1, using a similar duality and Corollary 6 of Wakker and Yang (2019).

**Theorem 7.** *Under Assumption 1, convexity of $\succcurlyeq$ is equivalent to concavity of $w$, and concavity of $\succcurlyeq$ is equivalent to convexity of $w$.* □

The theorem did not need any restrictive assumption, continuity or otherwise. Similar dualities were exploited by Yaari (1987), Abdellaoui (2002), Abdellaoui and Wakker (2005), and Werner and Zank (2019).[5] The duality in Fig. 1 also illustrates that Quiggin's (1982) insight, that one should use differences rather than absolute levels of probability distortion functions, is the dual of the insight of the marginal utility revolution (Jevons, 1871; Menger, 1871; Walras, 1874), being that changes of utility, rather than absolute levels themselves, are basic.

The axiomatization in Theorem 7 of convexity of $w$ through a widely used preference condition is appealing. If there are only two nonindifferent outcomes, deviating from Assumption 1, then $w$ is only ordinal and can be any strictly increasing function, thus can always be convex but never needs to be. Hence, Theorem 7 has characterized convexity of $w$ as general as can be.

We present yet another surprising equivalence, combining Theorems 5 and 7 for the special case of Yaari's (1987) dual model (RDU with linear utility), i.e., Wang's (1996) premium principle. The theorem gives an appealing characterization of the last part of Theorem 4 of Wang (1996). Linear utility is also commonly assumed for coherent risk measures (Artzner et al., 1999).

**Corollary 8.** *Under Yaari's (1987) dual model (Assumption 1, with X a nonpoint interval I and U the identity), $\succcurlyeq$ is outcome-convex (outcome-concave) if and only if it is concave (convex).* □

Thus the conditions, concerning mixing in two different dimensions—"horizontal" and "vertical"—are not only each other's duals, but they are also logically equivalent here. Röell (1987 §1) discussed these conditions in Yaari's model, but was not aware of their equivalence, nor has anyone else been as yet.

## 5. Further implications for existing results on risk in the literature

Yaari (1987) considered the special case of RDU for risk with linear utility. He characterized convexity of $w$ through aversion to mean-preserving spreads, which is a special case of Chew et al.'s (1987) theorem. Quiggin (1993 §9.1) and Röell (1987) similarly derived this result for linear utility. As our Theorem 7 showed, convexity with respect to probabilistic mixing provides an appealing alternative condition. It would have fitted better with the affinity condition for outcome addition that Yaari (1987) used, and the affinity condition for outcome mixing that Röell (1987) used, to axiomatize RDU with linear utility.

A surprising application concerns Köszegi and Rabin's (2006) reference dependent model. Masatlioglu and Raymond (2016) showed how Köszegi & Rabin's choice-acclimating personal equilibrium (CPE) is a special case of RDU. Loss aversion in Köszegi & Rabin's model then holds if and only if the probability distortion function in the equivalent RDU model is convex. Masatlioglu & Raymond's Propositions 3 and 10 used Wakker's (1994) version of our Theorem 7 to characterize loss aversion. They wrote: "we were able to demonstrate a previously unknown relationship between loss aversion/loving behavior and attitudes toward mixing lotteries within the CPE framework" (p. 2792) and "our results allow us to bring 20 years of existing experimental evidence to bear on CPE" (p. 2773). They required monetary outcomes and continuous utility. Our Theorem 7 shows that those restrictions can be dropped, and that the result holds in full generality. Their Proposition 6 uses aversion to mean-preserving spreads to characterize

[5] Recognizing them in particular situations is nontrivial. Thus, while well acquainted with Yaari (1987), Wakker (1994 Theorem 24) and Wakker (2010 p. 192 footnote 8) did not recognize this duality.

concave utility and loss aversion. Our Theorem 5 shows that their mixture aversion would have provided an appealing alternative characterization.

## 6. Conclusion

We have provided completely general axiomatizations of strictly increasing concave/convex utility and probability distortion functions, using only basic preference conditions. Unlike preceding results in the literature, we do not need to presuppose any continuity (or differentiability), and the preference conditions used (concavity and convexity) are all basic and appealing. All the richness we need in our analysis is that all simple lotteries are available in the preference domain. We have thus provided the most appealing and most general characterizations of concavity and convexity of utility and probability distortion functions presently available.

## Declaration of competing interest

There is no competing interest.

## Appendix. Proofs and further literature for risk

**Proof of Lemma 3.** This proof uses advanced tools from Debreu and Koopmans (1982). Note that neither they nor we assumed continuity of utility. This rather follows as a corollary outside $max(I)$. An independent proof from scratch can be obtained from Wakker and Yang (2019 Corollary 6).

Assume the comoncone of the lemma. Write $\pi_j = w(p_1 + \cdots + p_j) - w(p_1 + \cdots + p_{j-1})$. We suppress probabilities.

Concavity of $U$ implies concavity and, hence, quasiconcavity of $\sum \pi_i U(x_i)$, which is the representing function on the comoncone. Convexity of $\succcurlyeq$ follows.

We next assume convexity of $\succcurlyeq$, and derive concavity of $U$. Assume, for contradiction, that $U$ is not concave at some point $\beta$. First assume $\beta \in int(I)$. For any $\beta' \in int(I)$, we have a nondegenerate two-dimensional additive representation on $\{(x_1, x_2, \ldots, x_2) : x_1 \geq \beta', x_2 \leq \beta'\}$. For any $\beta' < \beta$ the additive representation is not concave in its second coordinate $x_2$. By Debreu and Koopmans (1982), it must be in its first coordinate, i.e., $U$ must be strictly concave below $\beta'$. Hence, it must be so everywhere strictly below $\beta$. For any $\beta' > \beta$ the additive representation is not concave in its first coordinate $x_1$. By Debreu and Koopmans (1982), $U$ must be strictly concave above $\beta'$ and, hence, everywhere strictly above $\beta$. Thus, $U$ is nondifferentiable at $\beta$. Debreu and Koopmans (1982) define degrees of (non)concavity. The degree of nonconcavity at $\beta$ is infinite, implying, as Debreu & Kopmans show, an infinite degree of concavity everywhere else. This would imply non-Archimedean function values, and a contradiction has resulted.

Because $U$ is strictly increasing, it must also be concave at $min(I)$, if such exists. If $max(I)$ exists and $U$ is not concave there, then it must be discontinuous there. This implies an infinite degree of nonconcavity there, which in turn implies infinite concavity everywhere below, which cannot be. □

**Proof of Theorem 4.** Follows from Lemma 3. □

**Proof of Theorem 5.** This result can be derived from Wakker and Yang, 2019, Corollary 7). We give here an independent proof. We first assume the properties of $U$ and $w$, and derive convexity of $\succcurlyeq$. (See also Wakker and Yang, 2019, Lemma B1.) Consider $P = (p_1 : x_1, \ldots, p_n : x_n)$, $Q = (p_1 : y_1, \ldots, p_n : y_n)$, and their mixture $M = \lambda P + (1-\lambda)Q$. The lotteries need not come from the same comoncone. We may assume $\lambda x_1 + (1-\lambda)y_1 \geq \cdots \geq \lambda x_n + (1-\lambda)y_n$. Define $\pi_j = w(p_1 + \cdots + p_j) - w(p_1 + \cdots + p_{j-1})$, and define $EU(P) = \sum \pi_j U(x_j)$, $EU(Q) = \sum \pi_j U(y_j)$ and $EU(M) = \sum \pi_j U(\lambda x_j + (1-\lambda)y_j)$. We have $RDU(M) = EU(M)$, but similar equalities need not hold for $P$ and $Q$. $EU$ is an expected-utility type functional using the decision weights of $M$. We have, with the first inequality following from concavity of $U$ (and, hence, $EU$) and the second from convexity of $w$: $RDU(M) = EU(M) \geq \lambda EU(P) + (1-\lambda)EU(Y) \geq \lambda RDU(P) + (1-\lambda)RDU(Y)$. This shows that $RDU$ is concave and, hence, quasiconcave, implying convexity of $\succcurlyeq$.

We next assume convexity of $\succcurlyeq$. Concavity of $U$ follows from Lemma 3. Remains to show that $w$ is convex. We first show that

$$w(p) \leq 1 - w(1-p) \tag{8}$$

for all $p$. Because this is direct for $p = 0$ and $p = 1$, we consider a $0 < p < 1$. We focus on lotteries $(p : x_1, (1-p) : x_2)$ and suppress probabilities. The next reasoning closely follows Wakker and Yang (2019, Lemma B.2).

Take an outcome in $int(I)$, 0 wlog, at which the concave $U$ is differentiable. Wlog, $U(0) = 0$. We consider a small positive $\alpha$ tending to 0, with $o(\alpha)$, or $o_\alpha$ for short, the usual notation for a function with $lim_{\alpha \to 0} \frac{o_\alpha}{\alpha} = 0$. In other words, in first-order approximations $o_\alpha$ can be ignored. We write $\pi_1 = w(p), {\pi_2}' = w(1-p)$.

We have $\pi_1 > 0$ and ${\pi_2}' > 0$. Because of continuity of $U$ on $int(I)$ and differentiability at 0, we can obtain, for all $\alpha$ close to 0, the left indifference in

$$({\pi_2}'\alpha, 0) \sim (0, \pi_1\alpha + o_\alpha) \preccurlyeq (\mu{\pi_2}'\alpha, (1-\mu)(\pi_1\alpha + o_\alpha)). \tag{9}$$

The preference is discussed later. We compare two values: the $\mu, 1-\mu$ mixture of the RDU values (which are the same) of the left two lotteries and the RDU value of their $\mu, 1-\mu$ mixture, which is the right lottery. We take $\mu > 0$ so small that the left outcome $\mu{\pi_2}'\alpha$ in the mixture is below the right outcome. Informally, by local linearity, in a first-order approximation the only difference between the two values compared is that for the left value the left outcome ${\pi_2}'\alpha$ receives the highest-outcome decision weight $\pi_1$ whereas for the right value it receives the lowest-outcome decision weight $1 - {\pi_2}'$. Convexity of $\succcurlyeq$ implies the preference in Eq. (9), which implies $1 - {\pi_2}' \geq \pi_1$.

Formally, note that different appearances of $o_\alpha$ can designate different functions. Thus we can, for instance, write, for constants $k_1$ and $k_2$ independent of $\alpha$: $k_1 o_\alpha + k_2 o_\alpha = o_\alpha$. The following is most easily first read for linear utility, when all terms $o_\alpha$ are zero. Write $u' = U'(0)$; $\mu$ can be chosen independently of $\alpha$. Here is the comparison of the aforementioned two values: $\mu\pi_1 u'{\pi_2}'\alpha + o_\alpha + (1-\mu){\pi_2}'u'\pi_1\alpha + o_\alpha \leq (1-{\pi_2}')u'\mu{\pi_2}'\alpha + o_\alpha + {\pi_2}'u'(1-\mu)\pi_1\alpha + o_\alpha$. Dividing by $\mu u'{\pi_2}'\alpha$, we obtain $\pi_1 \leq 1 - {\pi_2}' + \frac{o_\alpha}{\alpha}$. Now $\pi_1 \leq 1 - {\pi_2}'$ follows.

We finally derive convexity of $w$. We have to show that

$$w(p+\epsilon) - w(p) \leq w(p+\delta+\epsilon) - w(p+\delta) \tag{10}$$

for all $0 \leq p$, $\epsilon > 0$, $\delta > 0$, $p + \epsilon + \delta \leq 1$. (This reasoning is similar to Wakker and Yang (2019 proof of Corollary 7).) For this, we fix outcomes $\gamma > \beta$ and $r = 1 - p - \delta - \epsilon$, and consider the set of lotteries $\{(p : \gamma, q_1 : y_1, \ldots, q_n : y_n, r : \beta)\}$. That is, $q_1 + \cdots + q_n = \epsilon + \delta$. This set is isomorphic to the set of lotteries

$$\{(\frac{q_1}{1-p-r} : y_1, \ldots, \frac{q_n}{1-p-r} : y_n)\} \tag{11}$$

over $I = [\beta, \gamma]$, and normalizing the original $RDU$ representation gives an RDU representation on this set. Applying Eq. (8) to that RDU representation gives Eq. (10) for the original representation. □

**Proof of Theorem 7.** By Eq. (6), convexity of $w$ implies convexity of the RDU functional and, hence, convexity of $\succcurlyeq$.

Next assume that $\succcurlyeq$ is convex. We show that $w$ is convex. Assume three fixed outcomes $x_1 \succ x_2 \succ x_3$, suppressed henceforth. We assume $U(x_1) = 1$, $U(x_3) = 0$. For any lottery $(p_1, p_2, p_3)$ we consider the ranks $q_1 = 0$, $q_2 = p_1$, and $q_3 = p_1 + p_2$, and denote it $(q_2, q_3)$. The RDU representation can be written as $w(p_1) + (w(p_1 + p_2) - w(p_1))U(x_2) = (1 - U(x_2))w(p_1) + w(p_1 + p_2)U(x_2) = (1 - U(x_2))w(q_2) + U(x_2)w(q_3)$. Wakker and Yang (2019, Corollary 6) now implies concavity of $w$, with details as follows. We interpret $q_2$ and $q_3$ as outcomes, assigned to two states of nature. This turns convexity into outcome-convexity. $U$ is a weighting function assigning weight $U(x_2)$ to the state yielding the best outcome $q_3$, and $w$ is the utility function. Wakker and Yang's (2019) Assumption 2 holds. Their Corollary 6 implies concavity of $w$.

The result for convex $w$ and concave $\succcurlyeq$ follows from the result just obtained by defining $U^* = -U$, $\succeq^* = \preceq$, and $w^* = 1 - w(1-p)$. If outcomes are monetary and monotonicity w.r.t. money is considered desirable, then outcomes can be multiplied by $-1$. □

Further literature on aversion to mean-preserving spreads. The equivalence of outcome-convexity with aversion to mean-preserving spreads (and its variations discussed below) holds only under RDU. In general, there is no logical relation between these conditions. Before discussing further details, we note that outcome-convexity has been studied in the literature only when an underlying state space was specified, but this is equivalent to our definition for simple lotteries. Under compact continuity, outcome-convexity does imply aversion to mean-preserving spreads (Chateauneuf and Lakhnati, 2007 Theorem 4.2; Dekel, 1989 Proposition 2). The resulting preferences have been studied for optimal insurances (Ghossoub, 2019). If convexity (w.r.t. probabilistic mixing) holds, a condition implied by aversion to mean-preserving spreads under continuity, then by Dekel (1989 Propositions 2 and 3), under weak continuity, aversion to mean-preserving spreads becomes equivalent to convexity.

Bommier et al. (2012 Result 3) considered a more-risk-averse than relation weaker than aversion to mean-preserving spreads, with distribution functions crossing once. They showed for linear $w$ (EU) that their condition is equivalent to concavity of $U$. They also showed for linear utility (see their proof on pp. 1638-1639) that their condition is equivalent to convexity of $w$. They provided, more generally, comparative results. Chew and Mao (1995, Theorem 2 and Table II) used a yet weaker elementary risk aversion condition, implied by our outcome-convexity, and showed, under RDU, that it holds if and only if $w$ is convex and $U$ is concave. They assumed Gateaux differentiability, which under RDU is equivalent to differentiability of $w$, and continuity. Hence, under the latter two assumptions, they provided an alternative way to obtain our Theorem 5. Ebert (2004, Theorem 2) used a progressive transfer property, equivalent to Chew and Mao (1995) elementary risk aversion, to characterize concavity of $U$ plus convexity of $w$. Importantly, he did not need differentiability of $U$, although he did assume continuity. He considered welfare theory where states are reinterpreted as people and probabilities $p_j$ reflect proportions of a population. He used extra structural richness in allowing for any arbitrary replication of any group ("event") in the population. Ghossoub and He (2021) studied alternative versions of risk aversion, taking probability and utility risk premiums rather than preferences as primitives. □

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