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Asset Liability Management for Pension Funds
A Multistage Chance Constrained Programming Approach

(Asset Liability Management voor Pensioenfondsen
Een meer-perioden optimalisatiemodel met kansrestricties)

Proefschrift

ter verkrijging van de graad van doctor
aan de Erasmus Universiteit Rotterdam,
op gezag van de Rector Magnificus

Prof.dr. P.W.C. Akkermans M.A.

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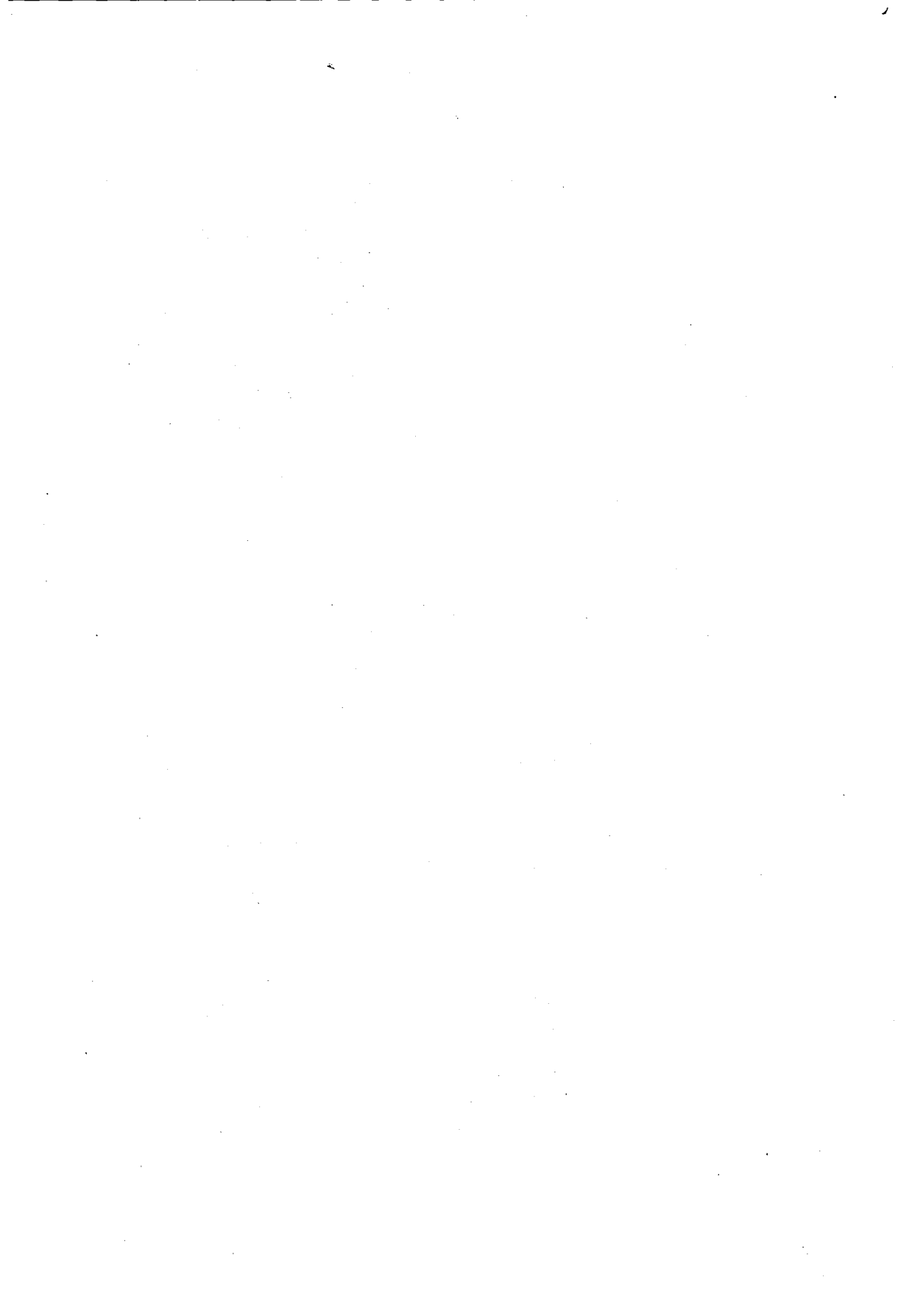


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To my parents



Preface

It has been nine years since I accepted Alexander Rinnooy Kan's proposal to set out on a Ph.D. study on a part time basis. Would I do it again? I think I would: business people are not that dull and academics are not so unworldly after all. Moreover, salary at the university is not that bad and neither is the commercial time pressure in industry.

I am greatly indebted to my supervisors, Alexander Rinnooy Kan and Guus Boender. Both of them have rather substantial obligations, other than advising on Ph.D. research. Nevertheless, they were always available to discuss ideas, read preliminary drafts of this thesis and to make formal arrangements with the Operations Research Department when necessary. Even though there have been extended periods of time during which I did not pay sufficient attention to my research, Alexander never seized the opportunity to quit. On the contrary, any reading material delivered by 2.00 am on Saturday would be commented and discussed by Monday at 10.00 am.

It would not have been possible to complete this thesis without the support of former employers and colleagues. At AKB, as well as at Pacific Investments, I was allowed to spend substantial time on research that was of no direct importance to them, even in times when business was difficult. In this respect I am especially grateful to Bob Out who granted me a sabbatical leave from Pacific Investments.

It has not been easy to initiate joint research on a subject that was not in the focus of interest of any department at the Erasmus University. This made the cooperation with Sjoerd Mosterd, Dennis Bams and Karin Aarssen all the more fruitful and enjoyable.

The computational results in this thesis would have been difficult to obtain without the support of Ortec Consultants, in particular from Fred Heemskerk and Jan van Mierlo, Jan Bisschop from Paragon Decision Technology, who provided me with a copy of the excellent AIMMS Modeling System, and Mn Services, the service organisation of the Pension Fund for the Metalworking, Pipe, Mechanical and Automotive Trades.

The importance of the indirect contributions of colleagues at the Operations Research Department, family and friends can hardly be overstated. Without them, there would be no point in the predominantly solitary task of preparing a dissertation.

Karin did not only contribute in an excellent way to the research on asset liability management: after she finished her studies, she abandoned the field of ALM, only to return as a new, overwhelming dimension to my life, which enriches it in every respect.

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Notation

General

The notion of time will be reflected by points in time t , $t = 0, \dots, T$. Period t , $t = 1, \dots, T$ refers to the span of time from $t-1$ to t . A scenario or a path through time will be defined as a consecutive sequence of states of the world. Each state has exactly one predecessor and may have many successors. The predecessor of t, s will be indexed by $t-1, \hat{s}$. The set I_t denotes all the information that is available at point in time t .

$[..]$	Closed interval
$(..)$	Open interval
$a_{ts}[..]$	Linear function that aggregates a sequence of cashflows to a lumpsum payment at point in time t in state s .
$c(.)$	Linear or piecewise linear cost function
$\exp(.)$	$e^{(.)}$
$O(.)$	Landau's O symbol
$Pr[.]$	Probability operator
$Pr[.,.]$	Conditional probability operator
$\phi(.)$	Cumulative density function of the standard normal distribution.
$\sigma[.,.]$	Covariance operator
$\ln(.)$	Natural logarithm
$N(\mu, \Sigma)$	Gaussian distribution with mean-vector μ and covariance matrix Σ .
I	Identity matrix
$E[.]$	Mathematical expectation operator
$Nodes(t)$	Set of nodes that were included in model $Twostage(t)$
$Intertemp(t)$	Set of nodes (τ, s) , $1 \leq \tau \leq t-2$ for which solvency requirements are enforced.

Superscripts

a	optimal solution to <i>Adjust</i>
f	short notation for a value which is an upper bound of a decision variable as well as a lower bound (i.e. $x^f = a \Leftrightarrow x^u = x^l = a$)
l	lower bound on decision variables

m	optimal solution to <i>MinAssets</i>
u	upper bound on decision variables
T	transpose of a matrix

Parameters

α	Demanded funding level
β_t	Maximal raise in contribution per period as a fraction of the cost of wages at point in time t
ϵ_{ts}	Realisation from random sampling
γ_{ts}	Discount factor for a cash flow at point in time t in state s
l_{ts}	Benefit payments and costs to the fund at point in time t in state s
g_{ts}	Continuous price inflation over period t in state s
h_{its}	Short notation for $\exp(r_{its})$
L_{ts}	Actuarial reserve at point in time t in state s
λ	Penalty parameter to penalise remedial contributions
λ_1, λ_2	Weighing coefficients
M_{ts}	Large constant at point in time t in state s
m_s	Weighting to determine importance sampling probability of state s
$n_{t,s}$	Number of states that succeed state (t,s) with positive probability
p_{ts}	Probability of state s at point in time t
q_{ts}	Importance sampling probability of state s at point in time t
r_{its}	Continuous return on investment i over period t in state s
$r(A)_{ts}$	Continuous return on investment portfolio A over period t in state s
$r(L)_{ts}$	Continuous growth of liabilities over period t in state s
R_{ts}	Vector of continuous returns on asset classes, price inflation, wage inflation and increase in gross national product over period t in state s
S_t	Number of states of the world at point in time t
Ω	First order autocorrelation matrix
w_{ts}	Continuous wage inflation over period t in state s
W_{ts}	Cost of wages over period t in state s

- ξ Random variable
 Ξ Probability space

Decision variables

- A_{ts}^+ Value of assets to invest at point in time t in state s in excess of the minimum required amount
 A_{ts} Total asset value before receiving regular contributions and making benefit payments at point in time t in state s
 B_{ts} Surplus at point in time t in state s
 C_{its} Amount of asset class i to sell at point in time t in state s
 D_{its} Amount of asset class i to buy at point in time t in state s
 Δy Annual raise in contribution as of time 0 as a fraction of the costs of wages
 f_{ts} Binary variable to register remedial contributions at point in time t in state s
 ψ_{ts} Probability of underfunding at point in time $t + 1$, given state of the world s at point in time t
 V Objective function value
 X_{its} Amount of money invested in asset class i at point in time t in state s
 X_{its}^+ Amount of money invested in asset class i at point in time t in state s in excess of the minimally required amount.
 x_{its} Fraction of asset value invested in asset class i at point in time t in state s
 Y_{ts} Regular contribution over period t in state s
 y_{ts} Regular contribution as a fraction of the cost of wages over period t in state s
 Z_{ts} Remedial contribution at point in time t in state s

Chapter 1

Introduction and Summary

This thesis presents a scenario based optimisation model to analyze the investment policy and funding policy for pension funds, taking into account the development of the liabilities in conjunction with the economic environment. Such a policy will be referred to as an **asset liability management (ALM) policy**.

The model has been developed to compute dynamic ALM policies that:

- guarantee an acceptably small probability of underfunding,
- guarantee sufficiently stable future contributions,
- minimise the present value of expected future contributions by the plan sponsors.

1.1 Problem Description

Pension Funds

A pension fund will be considered to be an institute that has been set the task to make benefit payments to people that have ended their active career. The payments to be made to the retirees must be in accordance with the benefit formulae that prescribe the flow of payments to which each participant in the fund is entitled. The word **participant** will be used to refer to all members of the pension fund: active members as well as inactive members.

In general, the pension fund has two sources to fund its liabilities: revenues from its asset portfolio (investment income and appreciation of the value of the portfolio) and contributions to the fund. Contributions are, by definition, made by the **sponsor** of the fund. The sponsor can be the employer, the active participants, or a combination thereof. Thus, at given points in time, the value of the assets of the fund is increased by receiving contributions and by appreciation of the value of invested assets and it is decreased by making benefit payments. It is the responsibility of the pension fund to balance this process in such a way that the fund meets the solvency standards in force, and that all benefit payments, now and in the future, can be made timely.

Important decisions that determine whether or not the pension fund will manage to fulfil its tasks are the **level of contributions** and the allocation of assets over asset classes in which the fund is willing to invest. This allocation is referred to as the **asset mix**.

These decisions cannot be made freely. The **level of contributions** has to be set in such a way that the sponsor of the fund is able and willing to pay them. This constraint is often reflected by a maximum level of contributions as a percentage of the costs of wages. Moreover, it is customary that annual hikes in contribution, again, as a percentage of the costs of wages, may not exceed a given level.

In principle, the fund is not restricted in its **choice of asset mix**. However, there are widely accepted perceptions of acceptable asset mixes which, in practice, result in upper and lower bounds on the percentage of assets to be invested in each asset category. Moreover, one has to heed constraints that are implied by the size and liquidity of the capital markets of interest, relative to the value of the securities that one would want to trade in a given period of time.

It depends on the ratio of income from contributions and revenues from the investment portfolio which decision, contribution level or asset allocation, is the more important one. In general, the higher the degree to which the pension fund has matured, i.e., the larger the percentage of participants who have ended their active career, the greater the relative impact of the investment decisions.

Although the way in which the **level of future benefit payments** will be determined is given by the benefit formulae, the actual level is uncertain. It is subject to the development of the characteristics of the participants which are determined by future career paths, life and death etc. The major source of uncertainty that affects the level of future benefit payments to be made by many Dutch pension funds, is the future development of price inflation and wage inflation: at retirement, the level of old age pension is usually 70% of the final salary. This pension includes a state pension to a fixed amount. It follows that pension rights of active participants that have been earned over past years of service will be increased by wage inflation. The benefits of inactive participants are often indexed with price inflation.

Once the value of assets proves to be insufficient to make benefits payments that are due, it is in general too late to take any measures to strengthen the financial position of the fund. To avoid this potential problem, the regulating authorities, in The Netherlands the Insurance Chamber have formulated **solvency requirements** for pension funds. They see to it that, at the end of each year, the pension fund has accumulated a level of assets that is sufficient to fund its liabilities.

It seems only natural to require the present value of assets to be at least as high as the present value of liabilities. However, the investment returns as well as the level of future

Stochastic
Continuity
analysis →

benefit payments are uncertain. As a consequence, it is unclear what the minimal present value of assets is that is sufficient to fund future benefit payments. Neither the present value of assets nor the present value of liabilities can be determined by a universally accepted method. In this monograph, the **assets** will be valued against their market prices. The valuation of liabilities is the domain of the actuary. Our ALM approach can be used in conjunction with any actuarial method of valuing liabilities. Nevertheless, to appreciate the problem of ALM, it is useful to have some background in actuarial principles. The **present value of liabilities** is usually determined by computing the present value of the expected future benefit payments. Given the characteristics of the current participants in the fund, the expected development of the characteristics (based on mortality tables, invalidity chances etc.) is computed. In conjunction with the benefit formulae, this development serves to compute the expected annual benefit payments for the planning period. Then, the present value of the liabilities can be obtained by discounting this flow of expected benefits. The discount rate that is used to compute the present value of the liabilities is often referred to as the **actuarial rate**. In The Netherlands, the annual actuarial rate that is commonly used to discount liabilities is 4%.

It is tempting to take a clear stand in the ongoing debate on the **appropriate level of the actuarial rate**. This discussion is frequently blurred by the fact that a substantial portion of the liabilities of Dutch pension funds stems from indexation of future benefits with price inflation and/or wage inflation. However, the indexation is usually conditional on the financial position of the pension fund. An actuarial rate equal to 4% can be considered high if it is used to discount indexed liabilities: one would have to realise an investment return equal to 4% + inflation, which would exceed an average of 8% annually over the past 50 years. On the other hand, if the benefit formulae do not contain any indexation promises, then a 4% discount rate seems to be rather low: over the past 50 years, an investor could easily have secured an average return on investments of 6%, without superb investment timing and without having to accept significant price risk or credit risk.

In the sequel, we shall not distinguish between conditional and unconditional liabilities. Liabilities will refer to the sum of **conditional and unconditional liabilities**. Thus, if the benefit formulae contain conditional indexation promises, our ALM approach will aim for a policy that enables one to make indexed benefit payments. As a consequence, one would expect that the minimum funding levels that follow from solutions to our ALM approach will generally exceed the minimum levels that are implied by solvency requirements which have been formulated solely on the basis of unconditional promises.

ALM Policy

A starting point for the analysis is the present state of a pension fund, defined by its actuarial and financial situation (asset value, premium reserve, level of benefit payments etc.), the benefit formulae and/or contribution formulae and the characteristics of the participants.

A good ALM strategy consists of investment decisions and decisions on the level of contributions that result in a desirable **risk/reward structure** with respect to the financial development of a pension plan. It minimises the cost of funding while safeguarding the pension fund's ability to meet its liabilities. The fund should be able to make all benefit payments timely, without becoming underfunded. Given these requirements, the present value of contributions to the fund should be minimised and contributions may be raised only modestly from one year to the next. Unfortunately, even an impeccable implementation of an excellent ALM policy cannot guarantee that all liabilities can be met under all circumstances. For example, when liabilities are indexed with inflation, exceptional situations may occur, in which inflation rates become so high that it is impossible to meet all liabilities, other than by raising contributions to a fantastic level. Since inflation rates can become very high over extended periods of time, one has to accept that there is a probability that the pension fund cannot meet its funding requirements. This probability is referred to as the **probability of underfunding**. To account for the fact that one cannot expect a pension fund to meet solvency standards under all circumstances, the solvency requirement has to be posed as a chance constraint. I.e., the ALM policy should ensure that the probability of becoming underfunded does not exceed a given level.

Neither asset mixes nor levels of contribution will be fixed for the entire planning period. Instead, decisions will be revisited when warranted by newly emerged circumstances, such as a changes in the funding level and altered perceptions of the future development of the world. However, stability requirements on the ALM policy may imply that one can only deviate so far from decisions that have been made in the past. These observations show that **current decisions and future decisions cannot be made independently**. Therefore, an ALM policy should consist of decisions to be made now and sequences of decisions to be made in the future. Future decisions should be conditional on the situation that has emerged at the time of decision making. Current decisions should anticipate on the ability to adjust decisions later on. Furthermore, to the extent to which they restrict choices in the future, they should reflect a correct trade-off of shorter term effects and longer term effects. Such a policy is referred to as a **dynamic policy**.

Defined Benefit Plans and Defined Contribution Plans

In the above description of the ALM problem, it has been assumed that the benefit formulae are given, whereas the contributions to the fund are to be determined. This is the case with **benefit defined** pension plans. In contrast with this type of pension plan, a **contribution defined** plan is characterised by fixed contribution formulae and uncertain benefit payments. Although the models and illustrations in this thesis assume a defined benefit pension plan, the approach that we present is also suited to determine investment policies for defined contribution plans.

1.2 Modelling an Uncertain Future by Scenarios

One of the central issues in ALM modelling, is the way in which uncertainty is modelled. Here, uncertainty will be modelled by a large number of scenarios, each of which reflects a plausible development of the environment within which ALM decisions have to be made. More specifically, future environments will be reflected by **states of the world**, which are defined by the level of actuarial reserve, the level of benefit payments, the level of costs of wages and the return on each of the asset classes over the previous period. These states of the world are independent of the decisions to be made with respect to asset mix and contribution policy. They are defined completely by factors that are **exogenous to the decision model**. A path through consecutive states of the world will be referred to as a **scenario**.

After generating a large set of scenarios, it is assumed that this set is a reasonable **representation of the uncertain future**: the model assumption is made that one of these paths will materialize. The uncertainty is still preserved in that the decision maker does not know yet which scenario describes the true future states of the world.

Scenario Structure

In order to model a multistage decision process with recourse, the states must be structured so that they can reflect the notion of time and the principle of information being revealed as time goes by. The desired information structure and the notion of time are ensured by imposing the tree shape scenario structure as depicted in Figure 1.

At point in time 0, there is only one state of the world: the state that can currently be observed. Given this state of the world there are many states of the world which could emerge by the end of period 1. Which one of them actually materializes will be known

only at time 1. In general, given state of the world s at time t , there are many states at time $t+1$ which succeed (t,s) with positive probability. This reflects the uncertainty regarding the future environment. At any point in time, the history by which the prevailing state of the world was reached is known: the scenarios are structured so that each node has a unique predecessor.

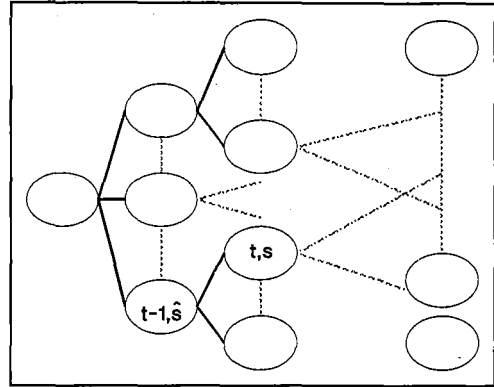


Figure 1 A Scenario Structure for ALM

Statistics of endogenous and exogenous state variables, such as the probability of underfunding and the expected surplus, play an important role in the ALM model. In order to compute these statistics, the scenarios have to be equipped with a **probability structure** on which the statistics can be defined. This structure should specify the probability of each state of the world to occur; unconditional, as well as conditional on the state of the world that has prevailed at the preceding point in time.

Consistency and Variety

The scenarios should be generated in such a way that future states of the world are consistent, i.e., stochastic and deterministic **relationships** between state variables **at each point in time** should be reflected correctly, subsequent states of the world should reflect the **intertemporal relationships** between state variables, and the **variety** of the states of the world should suffice to capture all future circumstances that one would want to reckon with. The scenario generator that is presented in chapter 4 satisfies these requirements. The ALM model that we propose, however, can be used in conjunction with any scenario generator that meets the requirements that have been specified above. For example, one could choose to employ a model that is based on economic theory, instead of the time series model that has been included in our scenario generator.

A noteworthy special case of reflecting sufficient uncertainty is the requirement that the scenarios may not allow for **arbitrage opportunities**. I.e., they may not include any states of the world in which it is possible to compose investment portfolios at price zero which have positive probability of a positive pay out, and which never have a negative pay out. In reality these opportunities will not occur to an extent that it is possible to exploit them systematically in an ALM policy. Therefore a realistic model should not

allow for arbitrage. Section 4.3.2 has been devoted to this subject. There, it is proven that the continuous probability distribution of states of the world that underlies our scenario generator does not allow for arbitrage opportunities. Moreover, for finite sample sizes, an algorithm is given that eliminates all arbitrage opportunities, if any, by extending a sample of given size by one well chosen state of the world.

1.3 The Position of our ALM Approach in the Literature

Chapter 2 contains an extensive discussion of publications on ALM for pension funds. Here, we shall restrict ourselves to a short characterisation of the main types of models, after which we shall position our approach relative to the existing methods.

One of the criteria that will be used to classify ALM approaches is whether or not the approach is dynamic. **Dynamic models** can be employed to compute policies that consist of actions to be taken now, and sequences of reactions to future developments. In contrast with dynamic models, **static models** do not make optimal use of the opportunity to react to future circumstances. Static decisions do not anticipate on the ability of making recourse decisions. As a consequence, the employment of static models may lead to:

- current decisions that do not reflect a correct trade-off between short term effects and longer term effects,
- current decisions that are extremely conservative because the ability to reduce risks in the future, when necessary, is neglected. This will cause the costs of funding to turn out unnecessarily high.

Still, most of the models that are currently being used for ALM decision support are static. This is probably caused by the fact that the computational effort to formulate and solve dynamic models for realistic problem sizes is large in comparison with static models. If computationally feasible, however, one should prefer a dynamic model.

Many ALM publications are based on **mean-variance analyses** of the surplus of a pension fund at a given horizon, taking into account stochastic liabilities. The trade-off between risk and reward, in this approach, is usually quantified as the trade-off between the expected level of the surplus at a given horizon and the standard deviation thereof. One of the main drawbacks of standard deviation as a measure of risk is that it does not distinguish between returns higher than expected and returns lower than expected. **Chance constrained programming** offers an alternate to quantifying risk by standard deviation which does not suffer from this shortcoming. One defines the probability that a certain

event will happen as a function of the model's decision variables. The probability of undesirable situations to occur can then be bounded by including constraints on the value of the associated statistics. To facilitate tractability, chance constrained models are usually presented in combination with the assumption that exogenous stochastic parameters, e.g., the growth of liabilities and investment returns, follow a probability distribution that is convenient from a computational point of view.

More recently published models on ALM are **stochastic programming models**. These models can be used to compute dynamic ALM strategies that are based on a set of scenarios which reflect the future circumstances that one wants to take into account. In principle, these scenarios can be based on any stochastic process that is considered to be appropriate to describe the environment for ALM decisions.

We propose a **mixed integer stochastic programming model**. It has the desirable properties of the aforementioned stochastic programming models in the sense that it can be employed to determine dynamic ALM policies that are based on scenarios, which can reflect any set of assumptions that one chooses to make on future circumstances. In contrast with the stochastic programming models that were mentioned earlier, our ALM model includes binary variables that enables one to count the number of times that a certain event happens. This possibility has been used to formulate chance constraints that are based on the probability distribution of states of the world that follows from the scenarios. In the case of ALM, this property is used to model and to restrict the probability of underfunding: at the planning horizon, as well as at intertemporal points in time. The choice has been made to sacrifice the ability to compute optimal solutions to problems of small sizes. Instead, we have opted for developing a heuristic by which good solutions can be computed to problems, the size of which suffices to model realistic problems. The main characteristics of the models that have been discussed in this section are presented in Table 1.

To conclude, let us summarize the properties which an ALM model should satisfy: The model should be suitable to determine a **dynamic ALM strategy**, consisting of an investment strategy and a contribution policy, which account for the development of liabilities. Decisions to be made now should anticipate on the ability to make state dependent decisions in the future. They should be the result of a **trade-off between short term effects and long term effects**. **Risk** must be reflected by the probability of underfunding and the magnitude of deficits when they occur. The model should accommodate the employment of realistic probability distributions of exogenous random variables, and, finally, the model should be feasible from a **computational** point of view.

Table 1. A Classification of ALM Approaches

approach	dynamic or static	explicitly consider probability of underfunding	realistic assumptions on probability distributions
mean-variance models	static	no	no
chance constrained programming	static	yes	no
stochastic programming	dynamic	no	yes
our approach	dynamic	yes	yes

To our knowledge, the ALM approach that is presented here, is the first one that meets all these requirements. Computational results, obtained on realistic problem instances, which are presented in summary in section 1.7, corroborate the theoretical notion that this type of model is superior to models that have been presented in the literature which do not meet all of the aforementioned requirements.

1.4 A Scenario Generator for Asset Liability Management

We have described the technical properties that the scenario structure should have, in order to serve as a framework within which dynamic ALM strategies can be analyzed and optimised. Let us now turn to the question as to what set of scenarios can serve as a reasonable representation of the future.

Different policy makers may consider different factors to be of interest to their ALM decisions. They may choose to base their policy on different assumptions and these assumptions should be reflected by the scenarios. Therefore, our ALM approach has been designed in such a way that it can be used in conjunction with any scenario generator that satisfies the conditions that have been stated in 1.2.

Figure 2 pictures the scenario generator that has been used to obtain the computational results that are reported in chapter 7. A **time series model** is employed to generate **future developments of price inflation, wage inflation and returns on stocks, bonds, cash and real estate** in such a way that means, standard deviations, autocorrelations and cross correlations between state variables are consistent with historical patterns.

Given the benefit formulae and all relevant data on the participants (e.g. civil status, age,

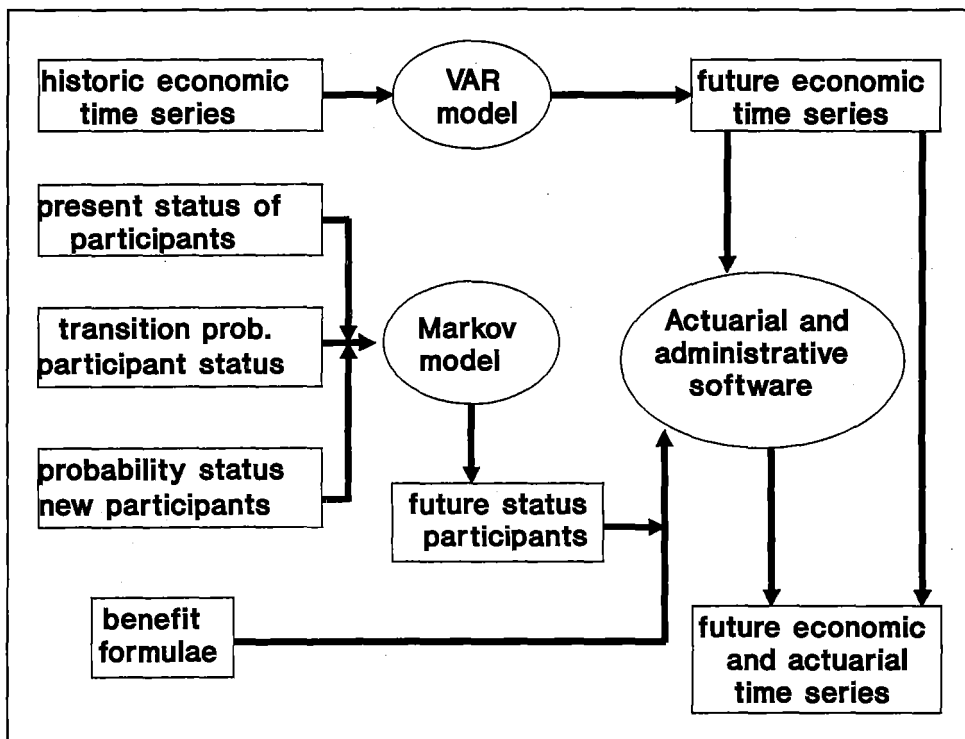


Figure 2 A Scenario Generator for ALM

gender, salary, earned pension rights, medical status, social status), a **Markov model** is employed to determine the future development of each individual that currently participates in the pension fund. For an employee, for example, it is determined whether he remains alive, retires, resigns, gets disabled and/or is promoted to another job category on an annual basis. These transitions are determined by probabilities which depend on characteristics of the individuals such as age, gender and employee-category. Additional promotions and the recruitment of **new employees** are determined in line with the intended personnel policy.

Given the development of wage inflation, the career of each employee in each future state of the world and the current reward system, the **cost of salaries, the level of benefit payments and the actuarial value of the liabilities** can be computed for each state of the world.

All information to describe states of the world is now available: investment returns on all asset classes have been obtained from the time series model, the administrative software

has generated the corresponding cost of wages and, to conclude, the actuarial software has provided the corresponding levels of benefit payments and actuarial reserves. Notice, that the scenario generator has been structured in such a way, that consistency between state variables within a state, as well as consistency between states of the world is preserved.

Once the scenarios have been generated, the following information is available for each state of the world:

- the level of benefit payments,
- the level of the actuarial reserve,
- the level of the costs of wages,
- the return on each of the asset classes over the preceding period of time.

Furthermore, the scenario structure implies that it is also known for each state

- what the preceding state of the world has been,
- which states of the world are possible successors, and what the probability is that they will emerge, given the current state.

All information that is contained in the scenarios is independent of ALM decisions. They are the subject of the next section.

1.5 A Dynamic Optimisation Model for Asset Liability Management

Chapter 3 presents an optimisation model that determines an ALM policy that consists of an asset mix and a contribution level for each state of the world. These decisions also determine the level of asset value and, in combination with the exogenously given level of liabilities, the funding level in each state of the world. The decisions in all states of the world are made simultaneously. This allows for a trade-off between longer term effects and shorter term effects, as well as for a trade-off between the outcome of decisions in different future states of the world.

The model has been developed to compute dynamic ALM policies that:

- guarantee an acceptably small probability of underfunding,
- guarantee sufficiently stable future contributions,
- minimise the present value of expected future contributions by the plan sponsors.

Because the probability of underfunding is an important concept in ALM and because it can be modelled in many ways which have substantial implications for its interpretation, we shall discuss it at more length in the following paragraph.

The Probability of Underfunding

The probability of underfunding has been defined on the set of scenarios. For example, suppose that there are 100 states of the world, each of which succeeds a given state of the world with probability 1/100, then the probability of underfunding, when starting from the given state of the world is equal to 1/100 times the number of succeeding states in which underfunding occurs. In general, if a maximum probability of underfunding equal to ψ^u is considered to be acceptable, then this is reflected by constraints which ensure that for each state of the world, the probability of being succeeded by a state in which underfunding occurs, is less than or equal to ψ^u . The probability of underfunding has been modelled in such a way that:

1. The model can account for **any probability distribution** that can be reflected by the scenarios. That includes distributions that are specified implicitly, such as the distribution of liabilities which may be given by benefit formulae in the form of computer programmes.
2. Probabilities of underfunding are **endogenous** to the model.
3. Probabilities of underfunding are taken into account **explicitly**, at **intertemporal** points in time, as well as at the **planning horizon**.

Underfunding

What would happen when a situation of underfunding occurs? It is not clear what would happen in practice. In our model, however, it will be assumed that a remedial payment is made which is precisely sufficient to **restore the required funding level**. The remedial contributions are included in the costs of funding. Thus, the probability of underfunding, as well as the magnitude of deficits when they occur are taken into account. The structure of the model can accommodate other assumptions with respect to measures to be taken in situations of underfunding as well. Alternative reactions that can be accommodated include remedial contributions to be made during a prespecified number of years until the desired funding level has been restored and, entirely or partially, failing to meet conditional indexation promises.

In summary, the ALM model that will be presented in chapter 3 can be used to compute ALM strategies which specify investment decisions and contribution levels to be set under a wide range of future circumstances. The decisions are made in such a way that the present value of expected contributions to the fund is minimal, subject to raising sufficiently stable annual contributions and the probability of underfunding at the end of each year being acceptably small when starting from the current situation, as well as from all future states of the world that the policy makers of the pension fund choose to take into account.

1.6 Computational Complexity

The proposed ALM model is a mixed integer linear problem, the size of which increases exponentially with the number decision moments. As a consequence, it is very difficult to solve the model to optimality for realistic problem sizes. Therefore, chapters 5 and 6 have been devoted to the development of a heuristic by which a good, but not necessarily optimal, solution to the ALM model can be obtained.

Chapter 5 presents a special case of the general scenario structure that has been presented earlier. Using this new structure, a heuristic can be used to compute good solutions to the ALM model. The heuristic consists of a backward procedure and a forward procedure. In the **backward procedure**, a sequence of two stage problems is solved; one for each point in time at which state dependent decisions can be made. The solutions to these problems serve to specify desirable situations of the pension fund in each state of the world. However, the two stage problems have not been formulated in such a way that it is always feasible to determine an ALM strategy that results in attaining the desirable situations in all states of the world. Therefore, the backward procedure is followed by a **forward procedure**. The latter consists of solving a one period model for each state world. Given decisions at preceding points in time, it minimises deviations from the desired situations that have been obtained from the backward procedure, subject to the constraints that the ALM policy should satisfy. The computational effort to solve the ALM model by means of the heuristic is proportional to the number decision moments.

The computational effort for each point in time is dependent on the number of states of the world that has to be taken into account. The fewer states of the world the smaller the computational effort to solve the models. Thus, the fewer the better. On the other hand, the number of states of the world should be sufficiently large to represent the underlying continuous probability distribution. In chapter 6, a **variance reduction technique**, importance sampling, will be employed to reduce the number of states of the world that is required to obtain a sufficiently accurate representation of underlying continuous probab-

ity distribution of states of the world.

1.7 Computational Experiments

Chapter 7 presents results of computational experiments with the ALM model. In order to obtain insight in the behaviour of the model on realistic problem instances, it has been applied to the data of a Dutch pension fund with an actuarial reserve in excess of 16 billion Dfl. and approximately 1,020,000 participants of which 240,000 are still in their active career.

One would expect ALM decisions for a wealthy pension fund to be different from those for a thinly funded pension fund. Therefore, three settings have been selected, which differ in the initial funding level and in the amount by which annual contributions may be raised from one year to the next:

Setting 1: a low initial funding level and a low maximum increase of contributions,

Setting 2: a high initial funding level and a high maximum increase of contributions,

Setting 3: the initial funding level to be determined by the ALM model in such a way that costs of funding are minimised subject to satisfying the solvency constraints with moderate maximum increases of contribution.

In all settings, the probability of underfunding was allowed to be at most 5% in each year. The cost figures in Table 2 and Table 3 are presented in mln. Dfl.

Table 2. Summary of computational results from the ALM model

Setting	Initial asset mix				Underfunding		PV Total costs
	Cash	Stocks	Property	Bonds	PV remedial contributions	Average probability	
1	66	21	13	0	699	10.6%	23,002
2	52	44	4	0	19	0.4%	25,356
3	0	100	0	0	42	0.6%	24,682

In order to compare the results that are shown in Table 2 to other approaches, static decision rules have been determined to specify time and state dependent contribution

Table 3. Summary of computational results from static decisions

Setting	Initial asset mix				Underfunding		PV Total costs
	Cash	Stocks	Property	Bonds	PV remedial contributions	Average probability	
1	49	18	33	0	24,340	21.8%	22,906
2	35	28	31	6	1276	2.3%	30,063
3	14	52	32	2	827	1.9%	27,099

levels, in combination with optimal static asset mixes. These results are presented in Table 3. As can be verified from the tables, the results from the dynamic ALM model are superior in all settings. In setting 1, because it does not violate solvency constraints as much as the static model. In settings 2 and 3 in which both models present feasible policies, the present value (PV) of the costs of funding is lower. Moreover, the present values of remedial contributions to be made when the static policy is pursued are 20 to 60 times as high as those that are associated with the dynamic policy from the ALM model. In order to assess the extent to which the results from the ALM model are due to its dynamic character, the results have been compared to results from a model that makes optimal time dependent and state dependent decisions, taking into account a horizon of one year. This comparison indicates that the results from the ALM model are largely determined by its dynamic character.

The computational results which are presented in more detail in chapter 7, provide the following insights with respect to the ALM approach presented in this monograph.

1. Dynamic ALM strategies lead to current decisions that are different from decisions to be made when following a static policies.
2. In comparison to the static models, the employment of the ALM model has resulted in strategies of which the costs of funding are lower, the probabilities of underfunding are substantially smaller and the magnitude of deficits, reflected by the costs of remedial contributions, has been reduced dramatically.
3. The favourable outcome of the comparison of policies determined by the ALM model with policies determined by static decisions, are to a major extent due to:
 - the fact that probabilities of underfunding at intertemporal points in time as well as at the planning horizon are endogenous to the model and have been modelled

explicitly, and

- the dynamic character of the ALM model which enables the policies to react to situations that have emerged at the time of decision making and to reflect a correct trade-off between their longer term effects and their short term effects.

Chapter 2

Asset Liability Management for Pension Funds, A Survey

2.1 Introduction

Asset liability management has drawn attention from academics and practitioners for several decades. Most of the literature focuses on techniques aimed at managing a portfolio of fixed income securities in such a way that the cash flow received from holding the portfolio in some sense matches the (projected) cash out flow of a stream of liabilities. Closely related is the extensive research published on asset liability management for banks, mainly concentrating on financial risks incurred from changes in interest rates. Recently published surveys on this field of interest can be found in Smink (1994) and Klaassen (1994).

In comparison, the literature on ALM for pension funds is rather modest although the recent flow of publications in this domain seems to indicate a growing interest from people with different backgrounds. Both actuarial and financial journals publish research on asset liability management for pension funds. Judging from the references, it seems that authors from one discipline are not always aware of the ALM literature published in journals from the other discipline. Non-members of the actuarial community can be referred to the debate among actuaries on the proposition that "This house believes that the contribution of actuaries to investment could be enhanced by the work of financial economists", Wilkie et al. (1993), for an enlightening exposition of the actuarial view on the merits and shortcomings of financial theory.

2.2 Model Classification

In chapter 3, it is shown that one of the key issues in modelling ALM concerns the way in which information is resolved. The available information at the moment of decision making can consist of data that are known with certainty, e.g. the premium reserve that an actuary would consider sufficient to cover all liabilities, and information of a stochastic nature, e.g. the perceived probability distribution of the return on stocks in the next decade.

The field of **stochastic programming** has developed techniques to model and solve problems in which information with a stochastic nature plays a dominant role. Since this is the case in the domain of ALM, we shall classify ALM models by means of a

classification from the stochastic programming literature, taken from Wets (1989a). The classification distinguishes between anticipative models and recourse models.

Anticipative Models

Those are models for which the decision does not depend in any way on future observations of the environment. These models are also referred to as static models. The present decision has to take into account all possible future environments since there is no opportunity to adapt decisions later on.

For example, imagine a wealthy pension fund which can afford to reduce contributions from 15% of the wages now to 2% of the wages next year, without serious risk of insolvency in the short term. Should it do so ?

Using an anticipative model, the answer would probably be negative: the model assumes that there will be no opportunity to increase contributions again if the tide turns. As a consequence, it would be too risky to lower contributions now.

If it is possible, contrary to the model assumptions, to react to future circumstances, then the use of an anticipative model may lead to overly conservative decisions.

Recourse Models

The recourse model allows for different decisions at different points in time which may be contingent on the state of the world at the time of decision making. The anticipative model is a special case of a recourse model. Decisions made in state (t, s) , $t=1, \dots, T-1$, $s=1, \dots, S_t$ are adaptive as well as anticipative. They are adaptive with respect to period t since they depend on information that has been revealed during this period. With respect to period $t+1$, however, the decisions are anticipative: they do not depend on observations at time $t+1$. A solution to a recourse model thus consists of a decision to be made now, and a sequence of decisions which is contingent on the prevailing states of the world. Such a solution will be referred to as a **dynamic policy** or strategy. Deciding simultaneously on present actions and future recourse actions allows for a trade-off between longer term effects and short term effects.

The knowledge that there will be recourse decisions in the future affects present decision making in several ways. In comparison with the anticipative model, one has greater freedom to pursue short term benefits because the decision to be taken now is not a once

and for all decision. The present decision does not only maximise short term benefits. It also reckons with the impact on the ability to respond to different circumstances in the future. Especially when it is conceivable that the environment alters rapidly in comparison with one's speed of response, it may be optimal to pay a price now for flexibility in the future.

Let us return to the example of the wealthy pension fund. Using a recourse model one would probably decide to lower contributions. The contributions would be reduced by as much as possible, subject to the existence of a feasible contribution policy in the future which ensures solvency of the fund, in the short term as well as in the longer term.

In general, a recourse model distinguishes between points in time and states of the world at which decisions can be made. Each decision utilizes all the information that is available at the moment of decision making. Anticipating on future decisions and recognizing the impact of present decisions on future decisions allow for an optimal trade-off between short term effects and long term effects of present decisions.

2.3 Anticipative Models for ALM

2.3.1 Mean-Variance Models

A considerable portion of the ALM publications is based on mean-variance analysis. This approach on which extensive literature can be found in financial text books (e.g. Haugen (1993), Elton & Gruber (1989) and journals, has been introduced by Markowitz. He considered the problem of composing a portfolio of securities such that the **expected return** on the portfolio would be maximal, given the level of risk that one is willing to accept. **Risk**, in this approach, is defined as the variance of the return about the mean. It is common practice to report the standard deviation instead of the variance because standard deviation can be expressed in the same units as expected return, which makes it an easier measure to interpret in a risk-return trade-off.

Most of the publications on ALM focus on determining an asset mix that goes well with a given set of liabilities. This problem definition reduces the ALM problem to making the right investment decision. It is only natural then to pose the ALM problem as a variant of the mean-variance investment problem. There appears to be a straight forward analogy. Where traditional portfolio theory concentrates on selecting a portfolio of marketable assets, ALM considers a portfolio which consists of two components: a portfolio of marketable assets and a given portfolio of liabilities. The choice of the asset portfolio determines the expected return on investments and the standard deviation thereof. The

expected return of the entire portfolio equals the expected return on investments less the expected growth of the liability portfolio. The important difference between an ordinary investment decision and an investment decision in an ALM context lies in the fact that the variance of the return on the whole portfolio is partly determined by the covariance between the return on the asset portfolio and the growth of the liability portfolio.

Various publications on this approach have stressed various aspects in the model formulation (e.g. Elton and Gruber (1988), Sherris (1992), Wise (1987a and 1987b)). The model formulation below is taken from Sharpe and Tint (1990). Although it is not the richest model, it does demonstrate the idea behind the mean-variance approach for ALM well.

MV

$$\text{Maximise } E[A_T] - E[L_T] - \lambda_1 \sigma^2[B_T] \quad (1)$$

s.t.

$$\sum_{i=1}^N X_{i0} = A_0 \quad (2)$$

$$E[A_T] = \sum_{i=1}^N E[h_i] X_{i0} \quad (3)$$

$$\sigma^2[A_T] = \sum_{i,j=1}^N X_{i0} \sigma[h_i, h_j] X_{j0} \quad (4)$$

$$\sigma[A_T, L_T] = \sum_{i=1}^N X_{i0} \sigma[h_i, e^{r(L)}] L_0 \quad (5)$$

$$\sigma^2[B_T] = \sigma^2[A_T] + \sigma^2[L_T] - 2\lambda_2 \sigma[A_T, L_T] \quad (6)$$

Given an asset value of A_0 at the beginning of the period it has to be decided how this money should be allocated over investment categories $1, \dots, N$. This is reflected by equation (2), the budget constraint. Next consider the objective function. Notice that for $\lambda_2 = 1$, in (6), the model reduces to a mean-variance model in terms of the ultimate surplus. Therefore some authors refer to these models as surplus optimisation models. A value for the parameter λ_1 can be chosen so as to reflect one's risk aversion. The objective function component $\sigma[A_T, L_T]$ denotes the covariance between the level of liabilities and the value of the assets at time T as a function of the asset mix. Sharpe and Tint have defined $-2\sigma[A_T, L_T]$ to be the **Liability Hedge Credit**. It measures the

contribution to the variance of the surplus due to the correlation between the return on the asset portfolio and the growth of the liabilities. In absence of liabilities ($L_T = 0$) or if the growth of liabilities is uncorrelated with each of the asset categories

($\sigma[h_i, e^{r(L)}] = 0, i = 1, \dots, N$), the Liability Hedge Credit is equal to zero. The parameter λ_2 has been introduced to enable the user of the model to indicate the importance of volatility of the asset value risk in relation to the Liability Hedge Credit.

Does standard deviation of the surplus reflect the investor's risk perception well ? Although **standard deviation** is a widely used risk measure, there have been many publications which point out the limitations of standard deviation as a risk measure, among them Markowitz (1959), Hagigi and Kluger (1987) and Arnott and Bernstein (1988). More recently, Sortino and Van der Meer (1991) have shown lucidly why standard deviation is an inappropriate risk measure for many investment situations. Their exposition seems to be particularly relevant for ALM strategies. One of the main drawbacks of the standard deviation as a measure of risk is that it does not distinguish between returns higher than expected and returns lower than expected. To the investor this difference is rather important.

2.3.2 Chance Constrained Models

Chance constrained programming offers an alternate to quantifying risk by standard deviation which does not suffer from this shortcoming. One defines the probability of a certain event to happen as a function of the model's decision variables. Then, the probability of undesirable situations to occur can be bounded by including constraints on the value of the associated statistics.

If one assumes that the returns on all asset categories and on the portfolio of liabilities are normally distributed¹ with a known vector of means and a known covariance matrix, then it is well known that the level of surplus at the end of the planning horizon is also normally distributed (see e.g. Kall (1976)). Several authors, among which Wilkie (1985), and Brocket, Charnes and Li Sun (1993), have used this property to formulate a chance constraint on insolvency:

¹ Notice that the normality assumption does not apply to the continuous returns. It applies to $(h_i - 1), i = 1, \dots, N$ and to $(e^{r(L)} - 1)$.

$$Pr[A_T \leq L_T] \leq \psi^u \Leftrightarrow E[A_T] - E[L_T] \geq \varphi^{-1}(\psi^u) \sigma[B_T] \quad (7)$$

For $\psi^u \in (0, 1/2)$ this constraint is equivalent to the following combination of a convex quadratic constraint and a linear constraint:

$$E[A_T] \geq E[L_T] \quad (8)$$

$$(E[A_T] - E[L_T])^2 \geq (\varphi^{-1}(\psi^u))^2 \sigma^2[B_T] \quad (9)$$

This type of chance constrained program was first suggested by Charnes and Cooper (1959). A chance constrained programming model for ALM can be obtained by combining equations (2), ..., (6), (8) and (9). In the remainder of this chapter we shall refer to this model as *CC*. In the financial literature, investment models which employ chance constraints can be found under several key words among which downside risk (Harlow (1991)), shortfall constraints (Leibowitz and Henrikson (1989), Leibowitz and Kogelman (1991), Leibowitz and Langetieg (1990)) and shortfall returns (Albrecht (1993, 1994)).

2.3.3 Chance Constrained Programming, Normally Distributed Returns and Mean-Variance Optimisation

Although the philosophy behind the mean-variance approach may be quite different from the idea underlying chance constrained programming, one could wonder to what extent their solutions would differ if all returns are assumed to be distributed normally. Let us compare solutions to *MV* with solutions to *CC*. Any optimal solution to *MV* has the property that a higher expected surplus is not attainable without accepting a higher variance of the surplus. This follows directly from the objective function. Instead of indicating one's risk aversion by specifying a value for λ_1 one could specify an upper bound on $\sigma^2[B_T]$ and maximize the expected surplus. An alternative formulation of *MV* could thus be obtained by replacing the objective by *maximise* $E[A_T - L_T]$ and imposing an upper bound on the variance of the ultimate surplus: $\sigma^2[B_T] \leq \sigma^2[B_T]^u$. The difference between *CC* and *MV* is now confined to the way in which risk is modelled. *MV* bounds the variance of the surplus whereas *CC* bounds the probability of insolvency. However, under the assumption of normally distributed returns, any relevant optimal solution to *MV* can be obtained as an optimal solution to *CC* and vice versa.

A graphical representation of the relationship of these models is given in the graphs below.

Mean - Variance Optimisation and Chance Constrained Programming

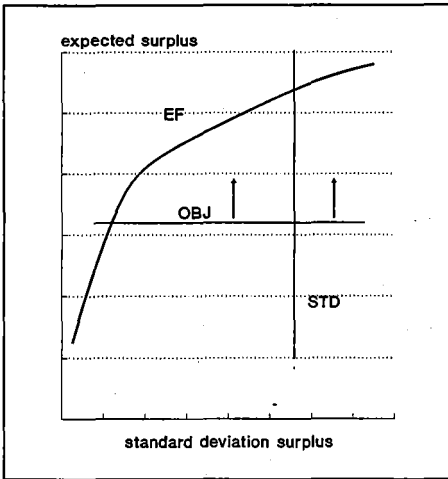


Figure 3

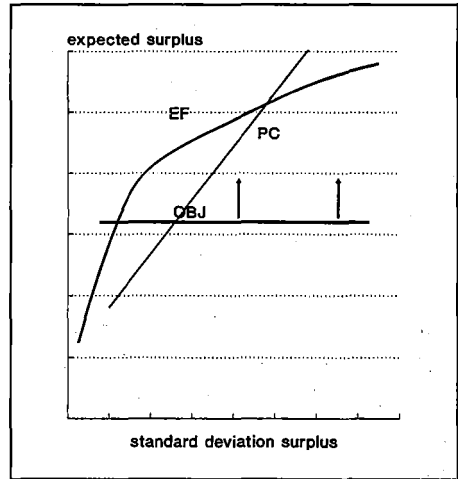


Figure 4

The curve EF reflects the mean standard deviation coordinates of the set of efficient solutions to MV and CC, i.e. portfolios with coordinates to right hand side of EF cannot be attained and portfolios with coordinates located to the left are dominated by portfolios on the curve EF. The vertical line STD reflects the risk constraint that is used in MV: the upper bound on the standard deviation of the ultimate surplus. Only portfolios with coordinates to the left hand side of STD are feasible. The objective of maximizing the expected surplus is reflected by shifting OBJ upward up to the point that is has only one point in common with the feasible region which is bounded by EF and STD. Thus the coordinates of the optimal portfolio can be seen to be defined by the intersection of EF and STD.

The line PC, defined by $E[A_T] - E[L_T] = \varphi^{-1}(\psi^\mu) \sigma[B_T]$ reflects the probabilistic constraint on insolvency. The gradient of the line PC is determined by the choice of a value for ψ^μ . Portfolios with coordinates located to the left of PC are feasible to CC. Portfolios with coordinates on the right hand side are not. As in MV, the coordinates of the optimal portfolio can be found by pushing up the line OBJ as far as possible while preserving feasibility. The optimal portfolio to CC is characterised by the coordinates of

the intersection of EF and PC.

Now consider a choice for the value of ψ^* so that the line PC goes through the intersection of EF and STD. Then the optimal solution to MV is also optimal to CC. Likewise, for any choice of ψ^* , the upperbound on the standard deviation can be specified so that the coordinates of the optimal solutions to the two models coincide. Thus, any optimal solution to one model can also be obtained as an optimal solution to the other model².

To summarize, in general probabilistic constraints on insolvency reflect the risk of the funding level dropping below a minimally required level. Since the standard deviation of the surplus does not distinguish between upside potential and downside risk, the probability of insolvency appears to be the better measure of risk. However, most publications on ALM models which utilize the probability of insolvency as a risk measure do so under the assumption of normally distributed returns. Due to the symmetry about the mean of the normal probability density function, the probability of a lower than expected return is always equal to the probability of a higher than expected return. As a consequence chance constrained programmes under the assumption of normally distributed returns distinguish between downside risk and upward potential no more than mean-variance models do.

Multiperiod Models

MV as well as CC takes a single period into account: from $t=0$ until $t=T$. It is not clear what value would be an appropriate choice for T . Pension funds typically have a long horizon, say 30 years. This would call for choosing T equal to 30 years. However, since the models discussed up to now do not include any requirements for t , $0 < t < T$, it may well be that the optimal solution to MV with $T = 30$ implies an unacceptably high risk of insolvency at one or more points in time t , $t \in [1, \dots, T-1]$. To illustrate this point, consider an asset mix such that the expected surplus at the horizon equals $E[B_T]$ with a standard deviation equal to $\sigma[B_T]$. If chance constraint (7) is binding, then it is likely that the probability of insolvency at time t , $t \in [1, \dots, T-1]$ is greater than ψ^* . To see why this is the case, assume that the annual returns on the surplus are distributed independently and identically. Then, the cumulative expected return decreases proportionally with t whereas the standard deviation decreases proportionally with \sqrt{t} . A more specific example would be the selection of an asset mix which largely consists of

² A formal exposition of the relationship between these models can be found in appendix A.

stocks. Assuming independently distributed annual returns in conjunction with a high expected return, such an asset mix may well seem to be attractive when judged by its long term perspectives. Nevertheless, it would not be recommendable to select such an asset mix, unless one can afford to sustain considerable losses in the shorter term.

In general, as is also argued in Elton and Gruber (1988), one should not only worry about the situation of the fund at the end of the period. Assuming that one has to satisfy a solvency requirement that is checked with a certain frequency, the model should only allow for solutions that would meet intertemporal requirements as well as end of period requirements. This calls for a multiperiod model.

Multiperiod models have been ^{veranschaulicht} proposed by several authors. Most publications which present multistage models employ some form of simulation to arrive at a solution of the model. Wilkie (1986a,b) simulates the behaviour of inflation and returns on various asset classes by means of a stochastic model which generates paths through time. These paths consist of consecutive states of the world defined by realisations from the stochastic process. Wilkie (1995) contains an extension of Wilkie (1986). It presents stochastic models to reflect returns on asset classes, using more advanced time series analyses. Hardy (1993) employs a stochastic simulation method, based on Wilkie (1986) to analyze a number of investment strategies for life offices. These strategies consist of simple decision rules which are applied at the end of each year for a period of 20 years. Hardy concludes that the stochastic simulation method provides the user with insights that could not be obtained from earlier studies in which these strategies were evaluated on a deterministic set of (worst case) scenarios. Moreover, the stochastic simulation led to the assessment that some of the strategies were unacceptably risky, whereas earlier studies indicated that all of the strategies were virtually riskfree.

Several authors use a single period mean-variance model in combination with a multiperiod simulation. It enables them to judge investment decisions obtained from a static single period model on its characteristics at the horizon T as well as at intertemporal points in time. Examples of this approach can be found in Booth and Ong (1994), inspired by the work of Sherris (1992), and Boender (1994). Unlike most ALM models, the one proposed in Boender (1994) does not aim for optimal mean-variance characteristics of the ultimate surplus. Assuming clean funding³, it determines an asset mix which minimises the volatility of contributions to the fund, subject to an upper bound on the average level of contributions. This variant of mean-variance optimisation is particularly relevant to policy

³ Under this policy ultimo period contributions are defined to be equal to the amount necessary to maintain the required funding level, $Y_t = \alpha L_t - A_t$.

makers who are predominantly interested in striving for a stable development of contributions.

2.4 Recourse Models for ALM

So far only static models have been considered. When using these models in practice, one would generally use them repetitively. After making decisions at time t , one updates the model with the latest observations at time $t+1$ and determines the contribution level and an asset mix at that time. This approach would be as good as a recourse model if decisions made at different points in time can be made independently of each other. However, this is usually not the case in ALM applications. In many cases there will be a constraint on the annual amount of buying and selling of securities. If this is not an issue of interest in itself, then it may play a role indirectly in the form of transaction costs. More important is the requirement of a stable development of annual contributions. This is one of the issues at the heart of a realistic ALM policy. Neither of these topics can be handled properly by employing a static model in a repetitive fashion.

Ziemba and Vickson (1975) already present dynamic models with financial applications. The need for dynamic models for ALM has been recognised by several authors in the field (see e.g. Booth and Ong (1994), Ludvik (1994), Sherris (1993), Janssen and Manca (1994)). Notwithstanding the notion that much is to be gained by employing recourse models to determine dynamic ALM strategies, only few models have been published which deal with this subject. Many of the proposed recourse models for investment decisions concentrate on portfolio optimisation without taking into account stochastic liabilities and funding decisions. These models are usually formulated as stochastic linear programmes. Kusy and Ziemba (1986) contains an application to ALM for banks. Golub et al (1993) and Holmer et al (1993) present stochastic programming models for fixed income investments. The latter contains computational results which corroborate the superiority of multistage models over single period models. Mulvey and Vladimirou (1988,1991) and Lustig, Mulvey and Carpenter (1990) report on quadratic multistage models for financial planning. These publications present computational results obtained from applying interior point methods and the progressive hedging algorithm which was proposed in Rockafellar and Wets (1991) and in Wets (1989b).

A two stage mean-variance model for ALM has been formulated in Mulvey (1994). It is not clear how the allocation of assets depends on the liabilities. This element of the system has not been published because of proprietary reasons. Cariño et al (1994) report on the development of the Russell Yasuda Kasai (RYK) model, an ALM model for a

Japanese insurance company. We shall discuss this model at more length since it can serve well to convey the basic idea of using stochastic linear programming for ALM to take stochastic liabilities into account. Consider the following model which is based on Cariño et al (1994)⁴.

MODIFIED RYK

$$\text{Maximise } \sum_s p_{t,s} A_{ts} - \sum_{s,t} p_{t,s} B_{ts} \quad (10)$$

s.t.

$$\sum_{i=1}^N X_{its} - A_{ts} = 0 \quad \forall t,s \quad (11)$$

$$A_{t+1,s} - \sum_{i=1}^N h_{i,t+1,s} X_{its} = Y_{t+1,s} - l_{t+1,s} \quad \forall t,s \quad (12)$$

$$B_{t+1,s} = -(L_{t+1,s} - L_{ts} - l_{t+1,s}) + Y_t + \sum_{i=1}^N (h_{it+1,s} - 1) X_{its} \quad \forall t,s \quad (13)$$

$$X_{its} \geq 0 \quad \forall i,t,s \quad (14)$$

The authors formulate a five period stochastic programming model, the solution of which consists of a dynamic investment policy which maximises the expected market value of assets at the horizon less expected costs of maintaining a non negative surplus. Notice that the surplus at time t can either be positive or negative. If it is negative, then $c(B_t)$ reflects costs, made to restore a non negative surplus. If the surplus is positive, then $c(B_t)$ reflects revenues to the insurer, which are received from withdrawing the surplus. The amount to be contributed or withdrawn is set equal to the surplus as defined in equation (13). The money under management has to be allocated over N investment classes subject to budget constraint (11). Constraint (12) reflects the accumulation of invested asset value over period t and cash flows from payments by the fund and to the fund at time t . Legal requirements which apply to the Japanese insurance industry and

⁴ The model in Cariño et al is geared more towards the Japanese Insurance business, whereas the model discussed here has been adapted to facilitate a discussion of its properties from the point of view of ALM for pension funds. The above model is a simplification in that no distinction is made between price return on investments and income from investments. In the original interpretation B_t is the net income shortfall. Loosely speaking this is the amount that the insurer has to contribute to the fund to satisfy solvency requirements. A negative value for B_t denotes a surplus which the insurer can withdraw from the fund. Furthermore Y_t denotes the inflow of deposits to the fund. As such it does not reflect costs to the insurer.

the role that the model was intended to play apparently called for a rather detailed description of cash flows. Modelling the flow of funds in such detail requires many decision variables at each time and state. In order to keep the size of the model down to tractable proportions, the number of scenarios has been kept small: to describe all relevant states of the world (in terms of the development of liabilities, interest rates for various types of fixed income investments and returns on several classes of equity investments) over 5 periods, adding up to a span of time of five years, the RYK model uses only 256 scenarios. The scenarios are generated by a Monte Carlo procedure under the assumption that the random variables are not autocorrelated. Next, the sample is corrected in such a way that all realisations satisfy prespecified upper and lower bounds and the sample variance of each of the random variables is equal to a desired variance which was determined on beforehand. It is interesting to see that the model concentrates on the reward component: it maximises the expected growth of net assets. The risk component is not explicitly included in the model. Neither the magnitude of payments to restore the required funding level at given times, nor the volatility of these payments or the probability of large deficits is constrained directly. Risk is accounted for by the variety of future environments over which the expectation in the objective function is computed. These future states of the world are largely determined by subjective assessments of appropriate bounds on the state variables. For a detailed description of the RYK model, as well as for an interesting comparison of computational results of different stochastic programming algorithms and commercial optimisation software, the reader can be referred to Cariño et al (1993).

2.5 Summary

In the preceding paragraphs the literature on ALM for pension funds has been discussed. A distinction has been made between anticipative models and recourse models, the latter being the more suitable type of model for ALM since this class of models allows for a more realistic reflection of the true decision making process.

Somehow or other each of the ALM models facilitates a risk-reward trade-off. The reward is usually linked to the magnitude of the expected surplus of the pension fund. A few models do include the funding policy as an endogenous variable. In these models the reward is usually linked to the cost of funding, the expected annual contributions or the volatility thereof. Most models include a notion of risk quantified by the standard deviation of the surplus, either directly or via a chance constraint on insolvency, coupled to the assumption of normally distributed returns.

The reflection of risk is one of the major issues in modelling ALM. Sortino and Van der

Meer (1991) have argued that appropriate risk measures for investment strategies should:

- explicitly consider the investor's goals,
- distinguish between true risk (uncertainty associated with undesirable events) and uncertainty (which can also be associated with outcomes that are better than expected),
- consider both the chance and the magnitude of adverse outcomes,
- be intuitively appealing.

It may not always be clear whether the latter requirement has been met. However, one would want an ALM model to reflect the notion of risk in accordance with the former three requirements.

One of the elements which plays a role in the above criteria is the probability of adverse outcomes. This assumes that it is possible to give an acceptably accurate assessment of this probability in the first place. Thus, the reflection of risk is not only determined by the variables and statistics by which an ALM strategy is analyzed. The assumptions with respect to the underlying stochastic process and the way in which this process is incorporated in the ALM decision model can also have a major impact on the outcome of the model. This issue will be treated more extensively in chapter 4 where the stochastic structure of our ALM model is discussed.

To conclude, let us summarize the properties which an ALM model should satisfy:

1. The model should be suitable to determine a dynamic ALM strategy, consisting of an investment strategy and a contribution policy, which account for the development of liabilities. Decisions to be made now should anticipate on the ability to make state dependent decisions in the future. They should be the result of a trade-off between short term effects and long term effects.
2. Risk must be modelled in accordance with the first three criteria as specified by Sortino and Van der Meer. This implies that the stochastic nature of the model should allow for computing the risk measures that they suggest.
3. It should be feasible from a computational point of view.

None of the models that has been discussed in this chapter satisfies all of these requirements.

Chapter 3

Modelling Asset Liability Management

This chapter presents a new optimisation model for ALM. The purpose of the model is to come up with an ALM policy, consisting of an investment policy and a funding policy for benefit defined pension plans⁵. Since the expense and benefit payments to be made are already defined (although the exact amounts to be paid are not known yet), the question becomes against what costs the plan can be funded. Of course, the lower the better; and therefore the reward component in our model is the present value of contributions to the fund. Risk in the context of ALM has to do with the possibility of failure to pay benefits and expenses whilst maintaining solvency. If the benefit formulae contain indexation promises, then they will be reflected by the level of benefit payments and the level of the actuarial reserves; not only when these promises are unconditional, but also when they are conditional. The model takes the probability of underfunding into account, as well as the cost of remedial contributions required to restore the desired funding level.

Section 3.2 develops a model formulation which aims at describing the ALM problem in mathematical terms without heeding the tractability requirement. This results in an infinite dimensional multistage stochastic optimisation problem for which no effective solution method exists. In section 3.3 the infinite dimensional optimisation problem will be reduced to a finite dimensional multistage stochastic programming problem through the use of a scenario representation of the uncertain future which should be taken into account. Although finite dimensional, the size of this model grows exponentially with the number of decision points. Consequently it is still too big to solve directly. Chapter 5 discusses the way in which this problem is handled. Before turning to the development of the model formulations, the next section discusses conditions which an ALM model should satisfy.

3.1 The Essential Elements of an Optimisation Model for Asset Liability Management

The concluding paragraph of chapter 2 lists the requirements which an ALM model should satisfy. This section goes into more detail regarding these requirements and their

⁵ The approach which we present is equally well suited to aid decision making for contribution defined plans. In case of a contribution defined one maximises future benefit payments given the contributions to the fund. In case of defined benefit plan one minimises the contributions given the defined benefit payments to be made. From a modelling point of view the difference is not an essential one. The reader can be referred to Ludvik (1994) for an interesting exposition of the similarity of the requirements to ALM policies for defined benefit plans and contribution defined plans.

consequences from a modelling point of view.

3.1.1 Elements Constituting a Dynamic Asset Liability Management Policy

The goal is to come up with a policy which minimizes the cost of funding while safeguarding the pension funds's ability to meet its liabilities. The fund should be able to make all benefit payments timely, without becoming underfunded. After all payments have been made, the value of the remaining assets should at least be equal to the minimum level which is required to fund future benefit payments according to the actuarial standards in force. The policy should consist of actions to be taken now, and sequences of reactions to future developments. Clearly none of the decisions may use information which is not available at the time of decision making. To make optimal use of the opportunity to react to future circumstances, decisions should anticipate on the ability of making recourse decisions.

Contribution Policy

One of the important instruments by which the pension fund can pursue its goals is the contribution policy. However, when setting contribution levels the fund will have to take into account the sponsor's ability and willingness to pay these contributions. It is questionable whether contributions can always be collected if they show unexpectedly steep annual increases. In order to accommodate the sponsor in making a multi-annual budget plan which includes realistic (upper bounds on) contributions to the pension fund, the funding policy should guarantee sufficiently stable annual contributions. More specifically we shall require the raise in annual contributions as a percentage of the costs of wages not to exceed a prespecified level.

Investment Policy

The second important instrument at the disposal of the pension fund is its investment policy. In this study we shall concentrate on strategic decisions of the investment strategy. More precisely, we consider the sequence of decisions regarding the allocation of assets over a limited number of asset classes. Such an allocation of assets will be referred to as an asset mix. Chapter 7 presents computational results based on a classification of investment classes which distinguishes between four investment categories: stocks, bonds, property and cash. The model should allow for more refined classifications as well. For example, it should be possible to make a further distinction by differentiating with respect to the maturity of security markets or geographical regions within the four classes which are mentioned above. On the other hand, the model need not be suitable to support

answering questions like 'Should I prefer Royal Dutch, or would I rather buy Exxon?'

The asset mix policy should be feasible to implement. This requires that proposed changes do not entail trading volumes which cannot be absorbed by the market without substantial impact on market prices. Furthermore there may be considerations exogenous to the model reflected by upper and lower bounds on proportions invested in one or more of the asset categories⁶.

Asset Liability Management Policy

A good investment policy and a sound contribution policy do not necessarily constitute an acceptable ALM policy. A good ALM strategy results in a desirable risk/reward structure with respect to the financial development of the pension plan. Such a policy comprises an investment policy and a contribution policy which take into account the development of the liabilities of the plan. In conjunction they should secure the desired risk/reward structure. In this monograph reward is quantified as cost of funding: the lower the better^{7,8}. Risk will be modelled in relation to the failure of the fund to meet solvency standards while making timely benefit payments. The next paragraph contains a more elaborate exposition of the requirements with respect to modelling risk and reward.

3.1.2 Risk and Reward

In the case of ALM risk and reward are two sides of the same coin. Therefore we shall discuss both risk and reward in this paragraph. The discussion is structured along the list of requirements on risk modelling which was given in paragraph 2.5.

Goals for Asset Liability Management

Pension funds adopt their right of existence from managing money that should serve to provide people with a pension after they retire. They are expected to be able to provide retirees with benefit payments. At this point, two types of pension plans have to be

⁶ Notice that constraints on trading volume in combination with constraints on asset proportions can quickly reduce a policy's feasible region. Especially when there are large differences between returns on different asset classes.

⁷ This interpretation of reward is formulated from the plan sponsor's point of view. In The Netherlands, pension plans are usually sponsored by employers or jointly by employees and employers.

⁸ Recall that we assume that the pension plan under consideration is benefit defined, thus there is no point in maximising benefit payments.

distinguished: **benefit defined plans** and **contribution defined plans**. The latter type of plan levies contributions, the level of which is determined by fixed contribution formulae. These formulae are typically exogenous to the pension fund's decision making. Given these contributions the pension fund should pursue an ALM policy aimed at maximising utility of present and future pensioners, defined as a function of benefits which they receive. This thesis focuses, however, predominantly on defined benefit plans. These are characterised by fixed formulae which specify the level of present and future benefit payments to be made. The benefits are typically defined as a function of one's salaries during active service. Usually the benefits are indexed with a combination of wage inflation and price inflation. Given the benefit formulae, the challenge becomes to devise an ALM strategy for providing the associated pension payments against minimal costs of funding. Pension funds typically accept the obligation to make indexed benefit payments for many years to come. Horizons longer than 30 years are commonplace. How should one value such a stream of future benefit payments? Posing this question lures one into the domain of the actuary. Using actuarial methods of computation and actuarial judgement, one (i.e., an actuary) can determine the value of current assets which is minimally required to fund future benefits. In the sequel we shall refer to this value as the actuarial reserve or the value of the remaining liabilities. Actuarial practice may not always be transparent to laymen. Nevertheless the actuarial reserve is a rather important figure: solvency requirements imposed by the regulating authorities demand pension funds to maintain a non-negative surplus, defined as the present value of assets less the actuarial reserve⁹.

In summary, an ALM policy should aim for a contribution policy and investment policy which enable the fund to make benefit payments in accordance with the benefit formulae, while maintaining the solvency of the pension fund. Given these requirements, the present value of contributions to the fund should be minimised and contributions may only be raised modestly from one year to the next.

The Distinction between Risk and Uncertainty

Uncertainty becomes manifest in investment returns, growth of liabilities and the development of the composition of participants to the fund. Realisations of these random variables are all reflected in the development of the surplus of the pension fund. Hence, uncertainty applies predominantly to the development of the fund's surplus.

⁹ Actually, there is no unique and widely accepted method to value current assets. We shall assume assets to be valued at their market price.

Risk is associated with adverse development of the surplus, i.e. with situations in which the surplus decreases to such a level that the desired level of funding cannot be sustained any more, or worse, that the ability of the fund to meet its liabilities comes at issue. This could happen in case of a rapid growth of liabilities in comparison with the growth of asset value.

Magnitude and Chance of Adverse Outcomes

As explained earlier, a negative surplus would be considered an adverse outcome. Thus an ALM model should take into account the probability of insolvency, as well as the magnitude of the deficit when it occurs.

3.1.3 Consequences for Modelling Asset Liability Management

The foregoing illustrates that the development of the level of asset values, the development of the minimally required actuarial reserve, the development of the annual cash inflows and cash outflows all play an important role in determining an ALM strategy.

None of these quantities is known with certainty: future asset values depend on uncertain investment returns, future liabilities depend on uncertain future participant compositions and may also depend on the uncertain development of macro-economic variables like price inflation and wage inflation. Many pension plans aim to provide indexed retirement payments. Under these circumstances, the development of the value of liabilities depends to a substantial extent on future price inflation and wage inflation¹⁰. The dominant question in ALM now becomes how to cope with the uncertainty regarding the values of these exogenous key variables. Ideally one would want to take all possible future states of the world (in terms of liabilities, cash flows and investment returns) into account, together with their probability of occurrence and derive an optimal decision for each possible state of the world. This is clearly not feasible. It is possible, however, to state some of the properties which the stochastic structure of the model should exhibit:

- Future states of the world that are taken into account should be consistent. I.e., stochastic and deterministic relationships between state variables at each point in time

¹⁰ Dutch pension funds usually do not guarantee to make indexed benefit payments. Instead, they often express their firm intention to do so, conditional on the financial position of the fund. In this thesis, no distinction is made between conditional and unconditional liabilities.

should be reflected correctly¹¹.

- Subsequent states of the world should reflect the intertemporal relationships between state variables.
- The variety of the states of the world should be sufficient to capture, implicitly or explicitly, all future circumstances that one would want to reckon with.

To the extent that relationships between state variables cannot be identified, it is important that the uncertainty with respect to their future values is sufficiently reflected by the variation in states of the world.

3.2 A Conceptual Asset Liability Management Model

Consider the following setting. A pension fund wants to determine an ALM policy for a span of time of T periods. Presently, at time 0, a contribution level for the first period and an asset mix at time 0 have to be decided on. At time t , $t = 1, \dots, T-1$, corrective actions may be taken. These decisions will be made, given all the available information at that point in time. This includes the financial state of the pension fund and the perception of the future development of the random state variables. To indicate the state dependency of decision variables at a given time they are denoted as functions of I_t .

A Mathematical Description of the ALM Process

Now let us develop a conceptual ALM model by following the chain of events that constitutes the ALM process.

At time 0 the contribution level for period 1 should be set. It should be between given minimum and maximum levels in dollar terms as well as relative to the costs of wages:

¹¹ In many cases the exact relationship between state variables may not be known. In these cases the future states of the world should be in accordance with the assumptions on which the decision makers want to base their policy. These assumptions can be drawn from different sources, e.g. from a panel of wise men, from statistical analysis of historic data etc.

$$Y_0^l \leq Y_0 \leq Y_0^u \quad (15)$$

$$y_0^l \leq \frac{Y_0}{W_0} \leq y_0^u \quad (16)$$

Benefit payments for period 1 should be made and the remaining wealth should be allocated over the asset classes, taking into account upper and lower bounds on the proportion to be invested in each class:

$$A_0 + Y_0 - l_0 = \sum_{i=1}^N X_{i0} \quad (17)$$

$$x_{i0}^l (A_0 + Y_0 - l_0) \leq X_{i0} \leq x_{i0}^u (A_0 + Y_0 - l_0) \quad (18)$$

At time t , $t = 1, \dots, T$ the asset value has been increased by the return on investments.

The asset value ultimo period t is defined as:

$$A_t(I_t) = \sum_{i=1}^N h_{it} X_{i,t-1}(I_{t-1}) \quad (19)$$

The solvency requirement dictates that this asset level should at least be equal to the required funding level times the value of the remaining liabilities:

$$A_t(I_t) \geq \alpha L_t \quad (20)$$

Including this constraint may be too restrictive: worst case realisations of L_t can be arbitrarily large, requiring arbitrarily large asset values, which in turn call for infinitely high investment returns or infinitely high contributions. It is the stochastic nature of the model that makes constraint (20) unrealistic. The solvency requirement can be included in a natural and consistent way by formulating it as a chance constraint:

$$Pr[A_t(I_t) < \alpha L_t | I_{t-1}] \leq \psi_{t-1}^u \quad (21)$$

Constraint (21) requires the probability of becoming underfunded at time t , given the situation at time $t-1$, to be less than or equal to ψ_{t-1}^u . It is not clear what would happen

in practice when a situation of underfunding occurs. The benefit formulae usually contain small print which enables the pension fund to reduce the liabilities by adjusting the benefit formulae. The fund could also call on the sponsor to make a remedial contribution to the fund. Yet another way to cope with the problem would be to accept the lower funding level for the time being and to initiate an ALM policy which is geared toward restoring the desired funding level at some point in the future. Of course, the latter solution would be possible only if there is no liquidity problem and if this is acceptable to the regulating authorities. In any case, it may be expected that measures will be taken to safeguard the existence of the pension fund. We shall assume that in case of underfunding a remedial payment is made which is precisely sufficient to restore the required funding level. To reflect this assumption constraint (19) should be replaced by:

$$A_t(I_t) = Z_t(I_t) + \sum_{i=1}^N h_{it} X_{i,t-1}(I_{t-1}) \quad (22)$$

And chance constraint (21) on the asset value dropping below the required level can now be replaced by a probabilistic constraint on $Z_t(I_t)$ taking on a positive value:

$$Pr[Z_t(I_t) > 0 | I_{t-1}] \leq \Psi_{t-1}^u \quad (23)$$

Now, still at time t , it should be decided to what percentage of the costs of wages the contribution for period t should be set, observing that it may not be raised by more than $100\beta_t\%$ percent points of the costs of wages:

$$\frac{Y_t(I_t)}{W_t(I_t)} - \frac{Y_{t-1}(I_{t-1})}{W_{t-1}(I_{t-1})} \leq \beta_t \quad (24)$$

After receiving the regular contributions and making benefit payments for period $t+1$ it has to be decided how to reallocate the assets,

$$A_t(I_t) + Y_t(I_t) - l_t = \sum_{i=1}^N X_{it}(I_t) \quad (25)$$

subject to upper bounds on trading volumes

$$h_{it}X_{i,t-1}(I_{t-1}) - C_{it}^u \leq X_{it}(I_t) \leq h_{it}X_{i,t-1} + D_{it}^u \quad (26)$$

and subject to upper -and lower bounds on the asset mix:

$$x_{it}^l(A_t(I_t) + Y_t(I_t) - L_t) \leq X_{it}(I_t) \leq x_{it}^u(A_t(I_t) + Y_t(I_t) - L_t) \quad (27)$$

The equations describing the ALM process have now been specified. When transaction costs are considered to be of importance, then the model can easily be extended to account for proportional transaction costs. Appendix B contains a model formulation which takes transaction costs into account explicitly.

Selecting an Objective Function for ALM

To reflect the goal of minimising the costs of providing the pension insurance the following objective is added to the model:

$$\text{Minimise } V = A_0 + Y_0 + \sum_{t=1}^{T-1} E[\gamma_t Y_t] + \lambda \sum_{t=1}^T E[\gamma_t Z_t] \quad (28)$$

The objective as specified above is to minimise the expected costs of funding. These costs consist of the current value of the assets, the expected present value of regular contributions to the fund and the expected present value of remedial contributions. In many cases current assets will be given. A_0 would then be a constant instead of a decision variable.

λ is a penalty parameter which reflects the preference of asking regular contributions over remedial contributions. Its value should be chosen so that an optimal solution will not allow for remedial contributions in excess of the minimal amount required to restore solvency. Such a value can be obtained as a function of γ and r . The discount factors γ serve to compute the present value of future cash flows. They reflect preferences with respect to the timing of contribution payments¹².

The choice of values for γ can have a major impact on solutions to the model. To

¹² Notice that the outcome of this trade-off also determines the extent to which the funding policy requires solidarity over time. If contributions are largely paid by the participants, then the choice for γ implies to which extent older participants are expected to pay for younger ones and vice versa.

illustrate this point, consider the following example.

Suppose that a fund has no obligations except for one benefit payment to the amount of 1 dollar, due after 1 year. Suppose that the return on investments equals $100r\%$. Minimising the cost of funding, subject to meeting the liability then corresponds

Minimise $y_0 + \gamma_1 y_1$ *s.t.* $(1+r)y_0 + y_1 = 1$. After eliminating y_0 the problem reduces to

minimise $\frac{1}{1+r} + (\gamma_1 - \frac{1}{1+r})y_1$. If $\gamma_1 - \frac{1}{1+r} \neq 0$ the problem is unbounded. The use of

different discount functions for the liability ($1/(1+r)$) and the contributions (γ_1) creates an arbitrage opportunity.

The example presents a deterministic environment in which there are no constraints which limit exploiting the arbitrage opportunity. The ALM model contains uncertainty with respect to the investment returns and the level of liabilities and sets of additional constraints which may prevent the problem from becoming unbounded. However, the ALM model incorporates this mechanism as well: discount factors higher than the return on investments will result in postponing contributions as much as possible, whereas discount factors lower than the return on the investments will lead to a solution where present contributions are maximised in order to make restitutions later on. This property reflects the choice between making financial investments (in excess of the minimally required amount) via the pensionfund and reserving the money to invest in business projects. Although this trade-off is perfectly valid, one should be aware of it and specify to what extent this arbitrage may be carried out. A natural upper bound on contributions to the fund follows from the maximum amount of money that is available with the sponsor to invest in the pension fund. This can be reflected in the model by specifying appropriate values for Y^u and y^u .

Several ways of determining γ are defensible. If γ_t is chosen so as to reflect the yield on a t -period zero coupon bond, then V can be interpreted as the amount of cash that would be needed now to fund all expected future contributions to the pension fund. One could also choose to set the value of the discount factors so that they reflect the cost of capital or the internal rate of return of the sponsor. This would be appropriate from a corporate finance point of view. The discount factors would then be consistent with the opportunity costs of capital which the sponsor invests in the pension fund instead of business projects. The above choices have in common that the future cash flows are treated as random variables whereas the discount factors are assumed to be known with certainty. One might argue that discount factors to be applied in the future are random

variables as much as future returns on bonds. To reflect that standpoint decisions to be made at time t should take the cost function into account that prevails at time t . The model allows for this option by specifying γ as a stochastic parameter rather than as a deterministic one.

3.3 Formulating a Dynamic Chance Constrained Program for ALM

The ALM model that was formulated in the preceding paragraphs may serve well to fix ideas on modelling ALM. Its practical relevance, however, is limited. It is an instance of an infinite dimensional programming model for which analytical solutions can be obtained only if one is willing to make simplifying assumptions to the extent that the model loses its practical relevance. Therefore we shall resort to the alternate of formulating a model that is tractable using numerical optimisation methods and captures the core of ALM. We shall assume that the probability distribution of the exogenous state variables can be reflected sufficiently well by a discrete distribution. As a consequence it suffices to consider a finite number of possible states of the world at each point in time. Solutions to the model will be obtained by using a method that does not require information on the probability distribution of the state variables, other than their realisations in given states of the world. Hence, the selection of state variables and the procedure to generate realisations of these random variables are not restricted by computational considerations. This opens up the possibility to use computer programmes which reflect complex relationships between state variables (e.g. the actuarial reserve given inflation figures, the benefit formulae and the status of participants to the fund) to generate internally consistent states of the world.

3.3.1 A Scenario Structure to Reflect an Uncertain Future

Chapter 4 contains a detailed description of the procedure which has been used to generate scenarios reflecting a discrete probability distribution of future states of the world. However, the model that is going to be presented in the next paragraph can be used in conjunction with any scenario generator that satisfies the conditions that are discussed in the remainder of this paragraph.

In the sequel we assume that the set of states of the world indexed by (t,s) , $t=0,\dots,T$, $s=1,\dots,S_t$, approximates the underlying continuous distribution sufficiently well to serve our purposes. I.e., they reflect all future circumstances that should be taken into account when determining an ALM strategy. In order to model a multistage decision process with recourse, these states should be structured so that they can reflect the notion of time and

the principle of information being revealed as time goes by. The probability of state (t,s) to occur is assumed to be known, as well as the probability of occurrence, conditional on the state of the world at time $t-1$.

States of the world are defined by the values of a set of exogenous variables. In our case the state of the world s at time t is defined by the realisations r_{its} , L_{ts} , l_{ts} and W_{ts} for $i=1,\dots,N$, $t=1,\dots,T$, $s=1,\dots,S_t$. A sequence of states of the world at consecutive points in time will be referred to as a **scenario**. The desired information structure and the notion of time are ensured by imposing the tree shape scenario structure as depicted in Figure 2.

At time 0 there is only one state of the world: the state that can currently be observed. Given this state of the world there are many states of the world which could emerge by the end of period 1. Which one of them actually materializes will be known only at time 1. In general, given state of the world s at time t , there are many states at time $t+1$ which succeed (t,s) with positive probability. This reflects the uncertainty regarding the future environment. At any point in time the history by which the prevailing state of the world was reached is known: node (t,s) has node $(t-1,\hat{s})$ as its unique predecessor. Statistics of endogenous and exogenous state variables such as the probability of remedial contributions and the expected surplus play an important role in the ALM model. Therefore, the scenarios should be equipped with a **probability structure** on which the statistics can be defined. For all (t,s) , $p_{t,s}$ and $Pr[(t,s)|(t-1,\hat{s})]$ are assumed to be computable. Furthermore it is assumed that $p_{s,t} > 0$ for $t=1,\dots,T$, $s=1,\dots,S_t$ and that

$$\sum_{s=1}^{S_t} p_{st} = 1 \text{ for } t=1,\dots,T.$$

The next paragraph presents the formulation of a dynamic chance constrained ALM model that is based on the scenario structure which was discussed in this paragraph.

3.3.2 A Multistage Chance Constrained ALM Model

The ALM model that will be presented in this paragraph can be derived as a special case from the conceptual ALM model that has been formulated in paragraph 3.2. The latter model was formulated without making assumptions with respect to the probability distributions of the random state variables. This generality makes it difficult to solve the model. Here, a model is proposed that assumes these distributions to be discrete.

Before formulating the discrete-state space model the formulation of the chance constraint will be reconsidered. Chance constraint (23) is essential to reflect the notion of risk.

However, in this form it is not possible to treat it directly as a constraint when solving the model. We shall now introduce binary variables to register states in which remedial contributions are made. Using these binary variables, a set of constraints will be derived which is equivalent to (23) under the assumption of discretely distributed state variables. Consider the following constraints:

$$Z_{ts} \leq f_{ts} M_{ts} \quad (29)$$

$$f_{ts} \in \{0, 1\} \quad (30)$$

If they are included then for any feasible solution $Z_{ts} > 0$ implies $f_{ts} = 1$. It follows that

$$\Psi_{t-1, \delta} \leq \sum_{s=1}^{S_t} Pr[(t, s)|(t-1, \delta)] f_{t,s} \quad \text{and thus any solution that satisfies constraints (29),}$$

(30) and

$$\sum_{s=1}^{S_t} Pr[(t, s)|(t-1, \delta)] f_{t,s} \leq \Psi_{t-1, \delta}^u \quad (31)$$

satisfies (23). Moreover, for any solution satisfying (23), there exists a set of values for f_{ts} so that (29), (30) and (31) are met. Hence (23) is equivalent to (29), (30) and (31).

Based on the ALM model that was presented in paragraph 3.2 and the above derivation of probabilistic constraints of underfunding, the following ALM model at page 44 can be formulated.

In the sequel this model will be referred to as the ALM model. Notice that this model is linear in the decision variables which is attractive from a computational point of view.

However, the size of the model increases exponentially with T . Chapters 5 and 6 concentrate on this computational issue.

MODEL ALM

$$\text{Minimise } V = A_{01} + \sum_{t=0}^{T-1} \sum_{s=1}^{S_t} p_{t,s} \gamma_{ts} Y_{ts} + \lambda \sum_{t=1}^T \sum_{s=1}^{S_t} p_{t,s} \gamma_{ts} Z_{ts} \quad (32)$$

s.t.

For $t=0, \dots, T-1, s=1, \dots, S_t$

$$Y_{ts}^l \leq Y_{ts} \leq Y_{ts}^u \quad (33)$$

$$y_{ts}^l \leq \frac{Y_{ts}}{W_{ts}} \leq y_{ts}^u \quad (34)$$

$$A_{ts} + Y_{ts} - l_{ts} = \sum_{i=1}^N X_{its} \quad (35)$$

$$x_{its}^l (A_{ts} + Y_{ts} - l_{ts}) \leq X_{its} \leq x_{its}^u (A_{ts} + Y_{ts} - l_{ts}) \quad (36)$$

For $t=1, \dots, T, s=1, \dots, S_t$

$$A_{ts} = Z_{ts} + \sum_{i=1}^N h_{its} X_{i,t-1,s} \quad (37)$$

$$h_{its} X_{i,t-1,s} - C_{its}^u \leq X_{its} \leq h_{its} X_{i,t-1,s} + D_{its}^u \quad (38)$$

$$A_{ts} \geq \alpha L_{ts} \quad (39)$$

$$Z_{ts} \leq f_{ts} M_{ts} \quad (40)$$

$$\frac{Y_{ts}}{W_{ts}} - \frac{Y_{t-1,\hat{s}}}{W_{t-1,\hat{s}}} \leq \beta_{ts} \quad (41)$$

$$\sum_{s=1}^{S_t} \Pr[(t,s)|(t-1,\hat{s})] f_{t,s} \leq \Psi_{t-1,\hat{s}}^u \quad (42)$$

$$f_{ts} \in \{0,1\} \quad (43)$$

Before turning to the subject of scenario generation we shall list the input which is required to formulate the ALM model. Three groups of exogenous parameters can be distinguished:

- the exogenous state variables l , L , r and W ,
- parameters to reflect the user's objective and requirements on the ALM policy, x^l , x^u , γ , Y^l , Y^u , y^l , y^u , C^u , D^u , ψ^u , α and β ,
- parameters for which the values are determined on computational grounds, M and λ . These follow from the values of the previously mentioned input as will be explained in chapter 7.

The next chapter treats the subject of scenario generation for ALM. It describes the scenario generator that has been used to provide for the input to the ALM model.

Chapter 4

Scenario Generation for Asset Liability Management

4.1 Introduction

The bulk of input required by the ALM model consists of scenarios which reflect the discrete probability distribution that has been selected to describe future states of the world. These scenarios form the basis for the ALM model.

Before turning to the discussion of the scenario generator, recall the following definitions which were given in paragraph 3.3.1: A **state of the world**, indexed by (t, s) , is defined by realisations $r_{it,s}$, $L_{it,s}$, $l_{it,s}$ and $W_{it,s}$. A path of consecutive states of the world is referred to as a **scenario**.

The scenarios provide the framework within which the ALM policy is analyzed. Clearly one would want this framework to be a **realistic and relevant** one. However, different policy makers may consider different factors to be of interest to their ALM decisions. They may choose to base their policy on different assumptions and these assumptions should be reflected by the scenarios. Therefore, a set of scenarios which is well suited to serve as a basis for ALM decision making for one policy maker may be a rather poor setting for another decision maker. Hence, it is of importance to pay sufficient attention to the selection of a scenario generator when using these models in practice.

The selection of an appropriate scenario generator is subjective, but only to some extent. Paragraph 3.3.1 already listed the conditions and assumptions with regard to the scenario structure which follow from its role in our ALM approach:

- each state of the world has many possible successors and precisely one predecessor,
- $p_{t,s}$ and $Pr[(t,s)|(t-1,\hat{s})]$ are computable, $p_{t,s} > 0$ and $\sum_{s=1}^{S_t} p_{t,s} = 1$.

Such a structure is reflected by the tree shaped scenario structure, which is pictured in Figure 1 at page 5, and serves as input to the ALM model. The remainder of this chapter is devoted to the procedure that has been used to obtain values for L, l, r and W in this scenario structure. Figure 2 pictures the scenario generator that was used to obtain the computational results in chapter 7. A time series model, more specifically a vector autoregressive model, is employed to generate plausible future developments of price

inflation, wage inflation and returns on stocks, bonds, cash and real estate. These time series are generated in such a way that means, standard deviations, autocorrelations and cross correlations between state variables are consistent with historical patterns.

To compute the corresponding values for the actuarial reserves, the level of benefit payments and the cost of salaries, the status of all participants in the fund and the benefit formulae are required. The development of the status of current and future participants is modelled by means of a Markov model which generates the future status of each participant to the fund, based on transition probabilities from one status to another. Next, these data serve as input to actuarial and administrative software which compute the associated levels of benefit payments, actuarial reserves and costs of salaries.

All information to describe states of the world is now available: investment returns on all asset classes have been obtained from time series model, the administrative software has generated the corresponding cost of wages and, to conclude, the actuarial software has provided for corresponding levels of benefit payments and actuarial reserves. As will be explained in paragraph 4.3.1 the probability of state (t,s) to emerge follows directly from the way in which the future economic times series are generated.

4.2 Requirements on Scenario Generation

Some of the requirements follow from the purpose that the scenarios serve in the ALM model. These concern the structure of the scenarios and the associated probability structure. These issues have been discussed in paragraph 3.3.1 and reviewed in the previous paragraph. This paragraph discusses additional conditions which the set of scenarios should satisfy. For an interesting survey regarding the merits and potential pitfalls of the use of scenarios to model an uncertain future environment, the reader can be referred to Bunn and Salo (1993).

Regardless of the decision procedure to be used, the scenarios should be internally consistent and they should be generated in accordance with the assumptions on which the decision makers wish to base their policy. The set of scenarios should reflect expected developments of the environment as well as the uncertainty with respect to the future environment. Notice that policies based on scenarios which do not reflect sufficient uncertainty may appear less risky than they actually are.

Policy assumptions to be made can be drawn from sources which are far apart from a methodological point of view. One could for instance rely upon a panel of experts, but

one could also choose to employ forecasts from an explanatory econometric model. One should be careful, however, not to generate scenarios that allow for arbitrage. The concept of arbitrage plays a central role in the theory of pricing derivative securities. Since the concept is also of importance to ALM we shall discuss it at more length.

Scenarios for Investment Returns and the Concept of No-Arbitrage

Under the assumption that there exists a riskless asset to invest in, financial theory defines an **arbitrage opportunity** to be a situation in which it is possible to buy a portfolio at price zero which pays out a positive amount with positive probability in the future, whereas the probability of negative pay outs equals zero (e.g. Ingersoll (1987)).

To illustrate the concept of no-arbitrage, consider the setting which is depicted in Figure 5 . It reflects an economy in which two securities are traded. At time zero, security 1 as well as security 2 is traded at a price of 1. At time 1, two states of the world can occur: with probability $1/3$ $p_1 = 2$ and $p_2 = 1.5$ and with probability $2/3$ $p_1 = p_2 = 0.5$. This set of scenarios reflects an arbitrage opportunity: at time 0 the price of a portfolio consisting of 1 unit of security 1 and -1 unit of security 2 is zero, however, at time 1 the portfolio will be worth either zero or 0.5, depending on which state of the world realizes.

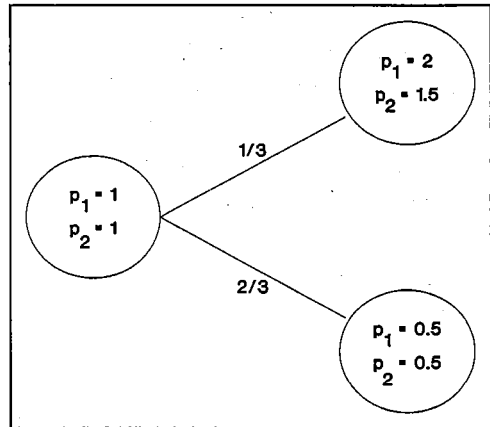


Figure 5. Example of Arbitrage Opportunity

Theoretical analyses usually presume that arbitrage opportunities cannot occur because investors would instantly exploit them, thereby affecting the security prices so that the arbitrage opportunities vanish: arbitrageurs would drive up the price of the cheaper portfolio by asking unlimited amounts while exerting selling pressure on the price of the more expensive portfolio. In practice, arbitrage opportunities do occur, but not with a frequency and to such an extent that a realistic ALM strategy could be based on them. Therefore, the scenarios should not contain arbitrage opportunities. We shall address this issue in section 4.3.2.

Pricing Derivative Securities and ALM

The concept of no-arbitrage is frequently used in financial literature to derive equilibrium prices for derivative securities. The assumption underlying this approach is that two investment strategies that generate identical future pay outs should have the same present value. Thus, to compute the price of a derivative security, i.e. a security of which the price is a function of one or more underlying values, such as an interest rate or a stock price, one designs a portfolio strategy which only trades in securities of which prices are known and which generates a pay out pattern that is identical to the pay out pattern of the derivative security that has to be priced. If such a strategy has been found, then, in an equilibrium situation, the price of the derivative should be equal to the price at which the portfolio strategy can be purchased.

Applying this line of reasoning to ALM, one could argue that the objective function value of an optimal solution to the ALM model is the present value of a portfolio strategy that generates a pay out pattern identical to the pattern of benefit payments. Thus, ALM could be viewed as a pricing problem of a complex derivative security: the level of benefit payments depends on underlying values such as the rate of inflation, the status of participants etc. Financial markets are incomplete in the sense that there is no (combination of) financial instruments by which the pay out pattern, as defined by the liability process can be replicated exactly. Moreover, given the present state of the art of financial theory, the process that determines the actual levels of future benefit payments and actuarial reserves is too complex to be described sufficiently accurate by a stochastic process that can serve to derive realistic dynamic ALM strategies analytically. In this thesis we shall not pursue the development of ALM from this viewpoint. Instead, we aim for a model that can be solved numerically to obtain useful ALM strategies. From the standpoint of option pricing, one might consider our ALM approach as a simulation based procedure to determine the price of a very complex derivative security.

For more extensive discussions of the concept of no-arbitrage and its role in financial theory the reader can be referred to textbooks on finance such as Duffie (1988) and Ingersoll (1987).

The following section presents a scenario generator for ALM which is based on the assumption that economic state variables can be described by a vector autoregressive process.

4.3 A Scenario Generation Model for Asset Liability Management

This section contains a detailed description of the scenario generator that was used to obtain the results in chapter 7. We shall subsequently discuss the components of the generator as depicted in Figure 2. The first part to be discussed is the model that generates investment returns and inflation rates, a vector autoregressive time series model. The second part concerns a Markov model, employed to describe the development of participants in the fund, which has been developed by Boender (1994). The remaining components of the scenario generator, administrative software to compute costs of salaries and actuarial software to calculate actuarial reserves and benefit payments, will not be discussed in this thesis. A discussion of actuarial principles which apply in The Netherlands can be found in Petersen (1992). Notice, however, that the scenario generator can be equipped with any module that provides for actuarial figures, given the economic state of the world and the status of the participants in the fund. More precisely, such a module should generate values for L_{ts} and l_{ts} that are consistent with realisations of the associated economic state variables R_{ts} .

4.3.1 A Vector Autoregressive Model to Generate Scenarios of Economic Variables

The first topic that will be discussed is the generation of plausible future time series of annual price inflation, wage inflation and total returns on the asset categories. Time series of economic states of the world will be generated in such a way that correlations, autocorrelations, variances and expectations of the elements that define the states of world converge to the corresponding statistics of observations that were made in the past. To accomplish this, a time series model is employed.

Time series models relate the value of variables at given points in time to values that these variables took on at preceding points in time. The difference between time series models and econometric models lies in the fact the latter type of models is usually based on economic theory whereas the former is not. Time series models are treated in many econometric textbooks (e.g. Judge et al (1985,1988)) and numerous publications. It is beyond the scope of this thesis to discuss the questions arising in model selection, estimation and testing in depth. This paragraph presents the main ideas underlying vector autoregressive models and briefly reviews methods to estimate their parameter values. It concludes with a stepwise description of the procedure by which future time series of economic variables are generated. For an extensive exposition of empirical vector autoregressive modelling the reader can be referred to Ooms (1993).

Vector Autoregressive (VAR) Models

Following the strong arguments of Sims (1980), we model the economic time series by means of a VAR model. Sims discusses the pros and cons of VAR models with respect to econometric models to support macroeconomic policy making. On methodological grounds as well as on technical grounds he argues that VAR models are more suitable to this end than econometric models. In addition to the theoretical arguments of Sims, when practising our ALM approach, the simple structure of a VAR model of order 1 allows for a straight forward interpretation of the model parameters. This provides for the opportunity to discuss results with policy makers who did not receive a quantitatively oriented education. When called for, coefficients of the VAR model can be adjusted in order to reflect subjective beliefs regarding future economic developments.

It is not uncommon in the financial theory to assume that asset prices follow a lognormal distribution¹³. In conformity with this assumption, we shall assume that the disturbances of the VAR model are distributed normally.

Autoregressive models can be used to describe linear relationships between the value of a variable at given times and the lagged values of this variable. Suppose, for instance, that annual rates of return on deposits tend to be high if they were high last year and that they tend to be low if they were low last year. More generally, suppose that the rate of return at given times can be explained partially by past rates of return. Such a relationship can be described by the following autoregressive model, allowing for autocorrelations up to the order K :

$$g_t = \rho_0 + \sum_{k=1}^K \rho_k g_{t-k} + u_t \quad u_t \sim N(0, \sigma^2) \quad (44)$$

To gain more insight in this model, consider the case that $K=1$: $g_t = \rho_0 + \rho_1 g_{t-1} + u_t$. For $|\rho_1| > 1$ g_t does not converge when $t \rightarrow \infty$. For $|\rho_1| < 1$ a mean reversion model remains¹⁴ with a long term mean equal to $\rho_0 / (1 - \rho_1)$. **Mean reversion models** are characterised by the property that the values of the explained variable show a tendency towards their long term average. The more a realisation deviates from the long term

¹³ Even though empirical evidence to the contrary has been reported (see e.g. Guimaraes, Kingsman and Taylor (1989) and Dert (1989)).

¹⁴ This type of model has been discussed extensively in the econometric literature (see e.g. Ooms (1993). and Rudebusch (1989)).

average, the stronger the tendency towards the average of the next value. Mathematically, $E[g_t | g_{t-1}]$ can be written as a convex combination of g_{t-1} and the long term average of g . To generalize this concept to the multivariate case, consider the situation that the inflation over a given year is correlated with the return on deposits over that year and with the rate of inflation over the previous year. Then, one would prefer a model that allows for cross correlation as well as serial correlation, as depicted in Figure 6, in which the arrows indicate statistical relationships. A VAR model does not include contemporaneous relationships that describe cross correlations directly. Instead, it allows for relationships between all variables at a given time and all lagged variables, as pictured in Figure 7. Whether or not a relationship is indeed included in the model depends on the value of the associated coefficients.

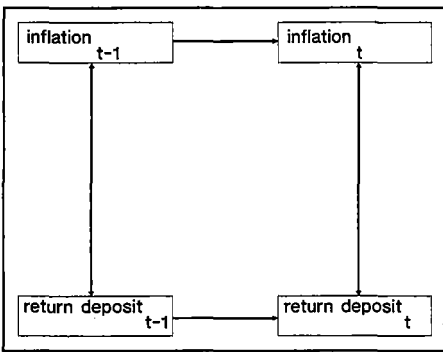


Figure 6 Cross correlation and first order serial correlation

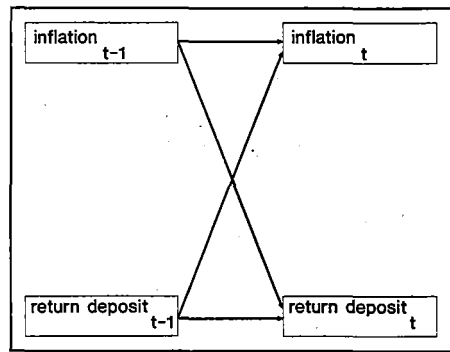


Figure 7 VAR correlations of the first order

The VAR model that was employed to describe the relationships between the continuous returns on asset classes, price inflation and wage inflation is given by:

$$R_t = \mu + \Omega (R_{t-1} - \mu) + u_t \quad u_t \sim N(0, \Sigma) \quad (45)$$

It allows for autocorrelation and auto cross correlation of the first order. Assuming that all eigenvalues of Ω are smaller than 1 in absolute value, $E[R_t]$ converges to μ for $t \rightarrow \infty$.

Estimating a Vector Auto Regressive Model for Economic Time Series

This paragraph addresses the problem of estimating the unknown parameters μ , Ω and Σ . Judge et al (1988) discusses several estimation procedures:

1. Ordinary Least Squares (OLS)
2. Seemingly Unrelated Regression (SUR) as introduced by Zellner (1962)
3. Maximum Likelihood (ML) by repeated SUR estimation

As is well known, the OLS estimator is the best linear unbiased estimator under the assumption that disturbances from different equations are uncorrelated and that disturbances are independently and identically distributed over time. OLS estimates of the coefficients of one equation are obtained by minimising the sum over time of the squared residuals. In case of a VAR model, values for all coefficients of the system of equations can be estimated by computing row wise OLS estimates, i.e. by performing OLS on each of the equations separately. This is equivalent to computing OLS estimators for the entire system simultaneously because, within the framework of OLS, covariances between disturbances from different equations are not taken into account; it is assumed that they are zero.

In the presence of correlation between disturbances from different equations at a given time, known as **contemporaneous correlation**, covariances between disturbances from different equations should be taken into account when estimating parameter values. In this case better estimates can be obtained through the use of SUR methods¹⁵. Zellner's SUR estimator is obtained by applying OLS, followed by a Generalised Least Squares (GLS) estimation. GLS estimates are obtained by minimising a quadratic function of the residuals from all equations simultaneously. The quadratic function, which is based on the covariance matrix of the disturbances, is chosen so that the variance of GLS estimates of the parameter values is minimal, given the covariance matrix of disturbances. In general this covariance matrix is not known and one uses an estimate thereof. Zellner uses a covariance matrix which is estimated from the OLS residuals.

After performing an OLS procedure, one can apply GLS in an iterative fashion: in each iteration, Σ is estimated from the residuals of the previous iteration. The newly estimated covariance matrix, in turn, is used to compute a new GLS estimate of μ and Ω . The estimates obtained from this iterative procedure converge to Maximum Likelihood estimates.

Although the asymptotic properties of the SUR estimators are superior to those of OLS estimators, it is not guaranteed that applying SUR methods on small samples will indeed produce better estimates than OLS. This is due to the fact that it is in general not possible

¹⁵ This is not the case if all of the equations have the same explanatory variables, then it can be shown that OLS estimates and SUR estimates are identical.

to obtain an unbiased estimate of the covariance matrix of the disturbances for finite numbers of observations¹⁶. There is no ready made recipe to decide which estimation method to choose. The smaller the sample size and the greater the difference between the number of explanatory variables, the better OLS. The greater the presence of contemporaneous correlation, the better SUR.

We have chosen to apply a stepwise ML method. Starting with an unrestricted ML estimation, an iterative procedure is carried out in which one non-significant parameter per iteration is removed from the model and new ML estimates are computed for the remaining coefficients until all insignificant coefficients have been removed. In case of more than one statistically insignificant parameter, the one to be fixed at zero is selected on statistical and economic grounds. The procedure is as follows:

- Step 1. Specify for each of the coefficients whether its value, based on economic theory, is expected to be negative, zero or positive.
- Step 2. Compute Maximum Likelihood estimates by applying GLS iteratively.
- Step 3. If all coefficients are statistically significant at the 1% level, STOP. Otherwise, select one of the non-significant coefficients to be fixed at zero, applying the following pick order:
 1. The least significant coefficient which was expected to be zero.
 2. The least significant coefficient which has a sign opposite to the expected sign.
 3. The least significant coefficient with a sign equal to its expected sign.

Go to Step 2.

A Vector Auto Regressive Model to Generate Economic Time Series

Given the estimated values of the parameters μ , Ω and Σ , (45) is applied iteratively to generate many scenarios of future time-series, structured as illustrated in Figure 1: At time 0 there is only one economic state of the world, defined by R_{0t} , the current values

¹⁶ This problem only arises when the number of explanatory variables per equation varies. Otherwise

$(1/(T-k)) \sum_{s,t=1}^T u_t u_s^T$ in which k denotes the number of explanatory variables in each equation is an unbiased estimator of Σ , given a set of T observations.

of the economic state variables. For $t = 1, \dots, T$, $s = 1, \dots, S_t$ the economic scenarios can now be generated by applying (45) repetitively:

$$R_{ts} = \mu + \Omega(R_{t-1, \hat{s}} - \mu) + \epsilon_{ts}$$

where

$$R_{ts} = \begin{bmatrix} r_{1ts} \\ \vdots \\ r_{Nts} \\ g_{ts} \\ w_{ts} \end{bmatrix} \quad \mu = \begin{bmatrix} \mu[r_1] \\ \vdots \\ \mu[r_N] \\ \mu[g] \\ \mu[w] \end{bmatrix} \quad \epsilon_{ts} = \begin{bmatrix} \epsilon[r_{1ts}] \\ \vdots \\ \epsilon[r_{Nts}] \\ \epsilon[g_{ts}] \\ \epsilon[w_{ts}] \end{bmatrix} \quad (46)$$

The values ϵ_{ts} are obtained by random sampling from $N(0, \Sigma)$. Thus, given state of the world (t, s') , all states $(t+1, s)$ with $\hat{s} = s'$ have positive probability to succeed $(t-1, s')$ and they are equally likely to do so¹⁷. States of the world (t, s) with $\hat{s} \neq s'$ have zero probability of succeeding $(t-1, s')$. More formally, if $n_{t, s'}$ states are generated as possible successors of state (t, s') , then

$$Pr[(t, s)|(t-1, s')] = \begin{cases} 0 & \hat{s} \neq s' \\ \frac{1}{n_{t, s'}} & \hat{s} = s' \end{cases} \quad (47)$$

Using the property of the scenario structure that each state has precisely one predecessor, $p_{t, s}$ for $t = 1, \dots, T$ can be computed by repeatedly applying the recursive formula $p_{t, s} = Pr[(t, s)|(t-1, \hat{s})]p_{t-1, \hat{s}}$.

A VAR Model to Specify Alternative Future Economic Patterns

¹⁷ Formally, this can be derived as follows. Let ξ_1, \dots, ξ_n be a random sample of size n . Choose indices m in such a way that $\xi_m < \xi_{m+1}$ for $m = 1, \dots, n-1$. Then the empirical cumulative density function based in this sample can be represented as (Bain and Engelhardt (1987)):

$$Pr[\xi \leq C] = \begin{cases} 0 & C < \xi_1 \\ \frac{m}{n} & \xi_m \leq C < \xi_{m+1} \\ 1 & C > \xi_n \end{cases}$$

Notice that (46) can also be used to generate scenarios that are based on parameter values other than those obtained from following the estimation method that we described in the previous paragraphs. This flexibility is of importance when one chooses to base an ALM policy on economic patterns which do not resemble historic patterns. When doing so, one could begin by specifying why future economic patterns are expected to be different from the ones that have been observed in the past. Given these motives, elements of μ , Ω and Σ ¹⁸ can be adjusted accordingly. Employing (46) then serves to structure the formulation of assumptions and to generate scenarios that are consistent with these assumptions.

Moreover, the estimated values of μ , Ω and Σ can serve as a starting point to specify parameter values that correspond to policy assumptions; it is an arduous task to specify parameter values without the frame of reference provided by estimates that are based on historic observations. A discussion on the asymptotic properties of the distribution of R when using different parameter values can be found in Boender and Romeijn (1991).

Extending the Scenario Generator to Account for Multiple Economic Regimes

Suppose that the historical time series cover periods of time with several economic regimes, then one could prefer to estimate one VAR model for each economic regime and to accommodate changes of regime in the scenario generator. That raises the question, however, how transition probabilities from one regime to another should be specified. Historical time series typically do not contain sufficient information to estimate them: either the number of transitions is too small, or the time series goes back so far that one should question its relevance for the specification of the future. One way out of this dilemma is to proceed as follows:

Given a small number of VAR models and the corresponding likelihood functions of states of the world (one model and one likelihood function for each economic regime) and the current state of the world:

step 1: compute the value of each of the likelihood functions,

step 2: specify a probability distribution of the current economic regime, based on the relative values of their likelihood function values,

¹⁸ If the perceived changes in the economic structure give rise to adjustments of elements of Σ , then it is recommendable to present Σ in a product form of standard deviations of disturbances per equation and correlations between these disturbances. It is easier to assess how these can be adjusted to be in accordance with policy assumptions than to adjust variances and covariances directly. Moreover, this approach enables one to preserve consistency between the elements of Σ so that it can still be interpreted as a covariance matrix.

step 3: for each possible state of the world that succeeds the current one:

- sample a current economic regime,
- generate the succeeding state by means of the corresponding VAR model, as has been described above.

Of course, as with the parameters of the VAR model, one can also choose to specify transition probabilities that are based on sources other than historic data and assumptions that come with the VAR model.

4.3.2 Arbitrage Free Scenarios

Does the scenario generator preclude arbitrage opportunities? In this section it will be shown that the scenario generator does not systematically generate arbitrage opportunities. Notwithstanding this desirable property it is conceivable that a finite sample of scenarios does allow for arbitrage. Therefore, this section also presents an algorithm to identify and eliminate arbitrage opportunities if there are any.

The Underlying Continuous Probability Distribution and Arbitrage

Given (46), the return on each of the asset classes is a lognormally distributed random variable. Thus, the return on each of the asset classes can take on any value greater than -100%. If we assume that all random variables are distributed independently, i.e. $i \neq j \Rightarrow \sigma[i, j] = 0$, then, the return on any investment portfolio equals a linear combination of independently lognormally distributed random variables. Hence, the portfolio has a positive probability of a negative return, as well as a positive probability of a positive return. As a consequence, no pair of portfolios exists, such that one portfolio outperforms the other portfolio with probability 1. It follows that there are no arbitrage opportunities if the state variables are an exact representation of the underlying continuous probability distribution. Fortunately, the assumption of independence is stronger than necessary. It suffices to preclude pathetic cases of correlation. More formally:

The continuous probability distribution which underlies the scenario generator precludes arbitrage if the conditional variance of the return on risky asset class j , given the returns on asset classes $i, i = 1, \dots, N, i \neq j$, is positive for all risky assets j .

Proof: Let X denote an arbitrage portfolio, i.e.

$$\sum_i X_i = 0 \quad (48)$$

$$Pr[\sum_i h_i X_i < 0] = 0 \quad (49)$$

$$Pr[\sum_i h_i X_i > 0] > 0 \quad (50)$$

(48) and (50) imply that there is at least one asset j , for which $X_j < 0$. As a consequence, (49) can be rewritten as:

$$Pr[\sum_{i,i \neq j} h_i X_i < -h_j X_j] = 0 \quad (51)$$

Suppose that j is not the riskfree asset¹⁹ and that r_j has a positive variance, given the realisations of $r_i, i \neq j$. Then h_j is lognormally distributed with positive variance. It follows that $Pr[h_j > M] > 0$ for any $M \in \mathbb{R}$. Hence

$$Pr[\sum_{i,i \neq j} h_i X_i < -h_j X_j | h_i, i \neq j] > 0 \quad (52)$$

for any set of realisations h_i , which contradicts (49) \square

Let the vector of continuous investment returns on risky assets r be distributed $N(\mu_r, \Sigma_r)$ and let Σ_r be non singular, then, for any $j \in \{1, \dots, N\}$, the conditional variance of r_j , given realisations $r_i, i \in \{1, \dots, N | i \neq j\}$, is positive.

Proof:

Define the following partitioning of r , μ_r and Σ_r :

$$r = \begin{bmatrix} r^1 \\ r^2 \end{bmatrix} \quad \mu_r = \begin{bmatrix} \mu^1 \\ \mu^2 \end{bmatrix} \quad \Sigma_r = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

It is well known that the distribution of r^1 given r^2 is normal with mean

¹⁹ If the riskfree asset is the only asset with a negative holding in the arbitrage portfolio, then it is easy to show that there also exists an arbitrage portfolio without a negative holding in the riskfree asset, under the assumption that the return on the riskfree asset is positive.

$\mu^1 + \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} r^2$ and covariance matrix $\Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$ (e.g. Taylor (1974)). Suppose that the conditional covariance matrix of r^1 equals zero, i.e. $\Sigma_{11} = \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$ and let

$q = \Sigma_{22}^{-1} \Sigma_{21}$. Then $\begin{bmatrix} \Sigma_{12} \\ \Sigma_{22} \end{bmatrix} q = \begin{bmatrix} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \\ \Sigma_{22} \Sigma_{22}^{-1} \Sigma_{21} \end{bmatrix} = \begin{bmatrix} \Sigma_{11} \\ \Sigma_{21} \end{bmatrix}$. Thus, we have shown that a

zero conditional covariance matrix of r^1 given r^2 implies that covariance matrix Σ_r is singular. Hence, a non singular covariance matrix Σ_r implies that the conditional covariance matrix of r^1 given r^2 is non zero \square

Using these results, non singularity of Σ_r implies that the probability density function that underlies (46) does not permit arbitrage opportunities if the covariance matrix of continuous returns on the risky assets has full rank. This implies that the continuous probability distribution which underlies the scenario generator does not permit arbitrage opportunities, unless there are asset classes of which the returns are perfectly correlated.

The Sample Distribution of Investment Returns and Arbitrage

However, since the ALM model only contains a finite number of scenarios, a sample might be drawn that does contain arbitrage opportunities. We shall show how one can ascertain whether or not a given set of scenarios allows for arbitrage. If it does, one can extend the sample by random sampling until the arbitrage opportunities have vanished (notice that the sample distribution of states of the world converges to the underlying continuous distribution as the sample size increases). A numerical example will be given to provide the reader with some intuition regarding the probability of generating arbitrage opportunities.

In practice, one may prefer to generate a set of scenarios of a given size in such a way that arbitrage opportunities are precluded in the first place. To accommodate this standpoint, we shall also present an algorithm by which the last scenario of a set of given size can be generated in such a way that all arbitrage opportunities, if any, are eliminated.

In the context of the economic scenarios which has been discussed in the previous paragraph, an arbitrage opportunity can be defined as the existence of a node (t, \hat{s}) such that solutions to *Arbitrage* can attain negative objective function values: constraint (54) requires the price of the arbitrage portfolio to be equal to zero. A non negative pay out in all future states of the world is enforced by (55). And, finally, $\sum_{i=1}^N (\sum_{s=1}^S h_{i,t+1,s}) X_{it\hat{s}}$ equals

the sum of the possible pay outs at the end of the period. If the objective function value is negative, then there must be at least one index s for which (55) is not binding and thus reflects a positive pay out at the end of the period.

Arbitrage

$$\text{Minimise } - \sum_{i=1}^N \left(\sum_{s=1}^S h_{i,t+1,s} \right) X_{it\bar{s}} \quad (53)$$

s. t.

$$\sum_{i=1}^N X_{it\bar{s}} = 0 \quad (54)$$

$$\sum_{i=1}^N h_{i,t+1,s} X_{it\bar{s}} \geq 0 \quad s = 1, \dots, S \quad (55)$$

If a solution to *Arbitrage* with a negative objective function value exists, then any rational investor who prefers more to less will want to buy unlimited amounts of the corresponding portfolio: the investor acquires the probability of receiving a payment in the future for free. Thus, the existence of a feasible portfolio X with a negative objective function value reflects an arbitrage opportunity.

Arbitrage Opportunities and Duality

One way to detect arbitrage opportunities, is to solve *Arbitrage*: if an opportunity exists, then its solution is unbounded, otherwise the optimal solution is 0. However, one can also use the dual program to check for arbitrage opportunities. The advantage of using the dual program is, that it also provides for a starting point to derive a method to eliminate all arbitrage opportunities by extending the sample by one appropriately chosen state of the world.

The dual program of *Arbitrage* reads:

Dual Arbitrage

$$\text{Maximise } 0\pi_0 + \sum_{s=1}^S 0\pi_s \quad (56)$$

$$\text{s. t. } \pi_0 + \sum_{s=1}^S h_{i,t+1,s} \pi_s = - \sum_{s=1}^S h_{i,t+1,s} \quad i = 1, \dots, N \quad (57)$$

$$\pi_s \geq 0 \quad s = 1, \dots, S \quad (58)$$

Applying the duality theorem of linear programming to *Arbitrage* and its dual gives:

If X is a feasible solution to *Arbitrage* and π is a feasible solution to *Dual Arbitrage*

$$\text{then } - \sum_{i=1}^N \left(\sum_{s=1}^S h_{i,t+1,s} \right) X_{it\bar{s}} \geq 0\pi_0 + \sum_{s=1}^S 0\pi_s = 0.$$

Hence, if there is a feasible solution to *Dual Arbitrage*, then there is no feasible solution to *Arbitrage* with a negative objective function value. It follows that any set of the states of the world which allows for a feasible solution π to *Dual Arbitrage* precludes arbitrage opportunities. The above argument is a generalisation of a similar argument for a special class of arbitrage opportunities which has been given in Ingersoll (1987).

Eliminating Arbitrage Opportunities

Consider *Dual Arbitrage* and suppose that there is no feasible solution π . Then, there may be arbitrage opportunities. To eliminate them, it would suffice to extend the set of possible future states of the world in such a way that the corresponding dual problem has feasible solutions. We shall show how to eliminate arbitrage opportunities by a well chosen extension of the sample of states of the world.

Intuitively speaking, arbitrage opportunities occur if it is possible to construct two portfolios of which one provides for higher returns than the other under all future circumstances. Such a situation can be eliminated by adding a state of the world in which the reversed situation occurs. Given a set of states of the world, arbitrage opportunities can be eliminated by extending this set with states in which portfolios that appeared to be superior, perform poorly and vice versa. As will be shown now, it takes only one additional state of the world to accomplish this.

More formally, given a set of S states of the world at time $t+1$, the question is how an additional state of the world $(t+1, S+1)$ can be specified which ensures the existence of a feasible solution to the dual program that is associated with states of the world $(t+1, 1), \dots, (t+1, S+1)$. In other words, how to determine $h_{i,t+1,S+1} > 0$ such that there exist values for π , $\pi_s \geq 0$ for $s=1, \dots, S+1$ that satisfy $\pi_0 + \sum_{s=1}^{S+1} h_{i,t+1,s} \pi_s = - \sum_{s=1}^{S+1} h_{i,t+1,s}$ $\forall i$.

Thus, values $h_{i,t+1,S+1}$ have to be chosen so that there are values for π that satisfy the following system of equations:

$$\pi_0 + \sum_{s=1}^{S+1} (1 + \pi_s) h_{i,t+1,s} = 0 \quad i = 1, \dots, N \quad (59)$$

$$h_{i,t+1,S+1} > 0 \quad i = 1, \dots, N \quad (60)$$

$$\pi_s \geq 0 \quad s = 1, \dots, S+1 \quad (61)$$

Rearranging (59) gives:

$$h_{i,t+1,S+1} = \frac{-\pi_0 - \sum_{s=1}^S (1 + \pi_s) h_{i,t+1,s}}{(1 + \pi_{S+1})} \quad (62)$$

Substituting $\pi'_0 = -\pi_0 / (1 + \pi_{S+1})$ and $\pi'_s = (1 + \pi_s) / (1 + \pi_{S+1})$ for $s=1, \dots, S$ in (61) and (62) leads to the following set of equations, which is equivalent to (59), ..., (61):

$$h_{i,t+1,S+1} = \pi'_0 - \sum_{s=1}^S \pi'_s h_{i,t+1,s} \quad i = 1, \dots, N \quad (63)$$

$$h_{i,t+1,S+1} > 0 \quad i = 1, \dots, N \quad (64)$$

$$\pi'_s > 0 \quad s = 1, \dots, S \quad (65)$$

As can be verified from (63), ..., (65), there are many values for $h_{i,t+1,S+1}$ and π which can serve to eliminate arbitrage opportunities. How could one select the best set of values for $h_{i,t+1,S+1}$ that precludes arbitrage? One may want to select values for $h_{i,t+1,S+1}$ in

such a way that they are in line with the underlying probability distribution. For instance, one could choose $R_{i,t+1,S+1}$ so as to maximise the likelihood function that follows from (46), subject to precluding arbitrage opportunities. This corresponds to the following problem in which σ_{ij} denotes the element at row i , column j of the covariance matrix Σ :

Max Likelihood

$$\text{Minimise } \epsilon, R_{i,t+1,S+1}, \pi' \sum_{i,j=1}^{N+2} \epsilon_i \sigma_{ij} \epsilon_j \quad (66)$$

s. t.

$$R_{i,t+1,S+1} = \mu_i + \sum_{j=1}^{N+2} \omega_{ij} (R_{i,t,S+1} - \mu_j) + \epsilon_i \quad i = 1, \dots, N \quad (67)$$

$$\ln[R_{i,t+1,S+1}] = \pi'_0 - \sum_{s=1}^S \pi'_s \ln[R_{i,t+1,S}] \quad i = 1, \dots, N \quad (68)$$

$$R_{i,t+1,S+1} > 1 \quad i = 1, \dots, N \quad (69)$$

$$\pi'_s > 0 \quad i = 1, \dots, N, \quad s = 1, \dots, S \quad (70)$$

Notwithstanding its intuitive appeal, *Max Likelihood* may prove difficult to solve, due to the nonlinear functions that appear in the constraints. Therefore, one may prefer to specify a model that requires less computational effort to solve and results in a reasonable additional state of the world that precludes arbitrage. For instance, one could avoid unnecessarily extreme values for $h_{i,t+1,S+1}$ by solving *Max Deviation*, which minimises the largest absolute deviation of $h_{i,t+1,S+1}$ from its expected value. Recall that $E[h_{i,t+1,S+1}]$ denotes the mathematical expectation of returns over period $t+1$ that follows from (46), it is independent of the decision variables $h_{i,t+1,S+1}$.

These results can be used to generate an arbitrage free set of economic scenarios.

One should be aware, however, the distribution of any scenario that has specifically been constructed so as to eliminate arbitrage opportunities, is different from, and possibly dependent on, the other scenarios. This may complicate statistical inference in a later stadium. For practical purposes, one may choose to neglect this. From a theoretical point of view one might prefer to generate the entire set of states of the world by random

sampling, checking for arbitrage opportunities, and extending the sample by random sampling until there are no arbitrage opportunities left.

Max Deviation

$$\text{Minimise } h_{i,t+1,S+1}, \Delta, \pi'_s \quad \Delta \tag{71}$$

s.t.

$$\Delta \geq \Delta_i \quad i = 1, \dots, N \tag{72}$$

$$\Delta_i \geq h_{i,t+1,S} - E[h_{i,t+1,S}] \quad i = 1, \dots, N \tag{73}$$

$$\Delta_i \geq -h_{i,t+1,S+1} + E[h_{i,t+1,S+1}] \quad i = 1, \dots, N \tag{74}$$

$$h_{i,t+1,S+1} = \pi'_0 - \sum_{s=1}^S \pi'_s h_{i,t+1,s} \quad i = 1, \dots, N \tag{75}$$

$$h_{i,t+1,S+1} > 0 \quad i = 1, \dots, N \tag{76}$$

$$\pi'_s > 0 \quad i = 1, \dots, N \quad s = 1, \dots, S \tag{77}$$

For sufficiently large sample sizes and realistic coefficients of the VAR model, it is unlikely that it is necessary to extend the sample substantially in order to eliminate arbitrage opportunities. To illustrate this point, consider the setting that there are only two assets to invest in. Suppose that their returns follow a bivariate normal distribution with mean μ and covariance matrix Σ ²⁰. Then, the difference of the returns is normally distributed with mean $\mu_1 - \mu_2$ and variance $\sigma^2[1] - 2\sigma[1, 2] + \sigma^2[2]$. Hence, the probability that the return on one asset exceeds the other can be obtained from the table of the standard normal distribution, after transforming the normal distribution of the difference of the two returns to the standard normal distribution. Let the return on the first asset exceed that of the second asset with probability p . Then, the probability that asset 1 outperforms asset 2 in all cases equals p^S . Likewise, the probability that asset 2 outperforms asset 1 under all circumstances is given by $(1-p)^S$. The probability of an arbitrage opportunity, given a random sample of size S , equals the probability that the first asset is superior plus the probability that the second asset is superior, i.e. $p^S + (1-p)^S$. Table 4 above presents a numerical example of the probability of creating arbitrage opportunities by random sampling as a function of the sample size. Of course,

²⁰ The following analysis also holds if the continuous returns are assumed to be distributed normally.

the example is a simplification of the process which underlies the scenario generator and the probability of generating arbitrage opportunities for given sample sizes may differ from the figures presented here. One should be particularly cautious when analyzing problems in higher dimensions. Nevertheless, the example can serve to provide the reader with some intuition for the probability of creating arbitrage opportunities by random sampling.

Table 4. The probability of creating arbitrage opportunities by random sampling

sample size	probability of asset 1 being superior	probability of asset 2 being superior	probability of an arbitrage opportunity
1	0.4013	0.5987	1.00
10	0.0001	0.0059	0.0060
20	$1.1732 * 10^{-8}$	$3.5009 * 10^{-5}$	less than 0.00004
40	$1.3765 * 10^{-16}$	$1.2257 * 10^{-9}$	less than 10^{-8}
70	$1.7492 * 10^{-28}$	$2.5389 * 10^{-16}$	less than 10^{-15}
100	$2.2229 * 10^{-40}$	$5.2593 * 10^{-23}$	less than 10^{-22}
$\mu_1 = 5, \mu_2 = 10, \sigma[1,2] = 0, \sigma[1] = 0, \sigma[2] = 20$			

4.3.3 Generating Future Time Series of Liabilities and Cost of Salaries

As explained before, the ALM model requires time-series of salaries, benefit payments, and actuarial reserves which are consistent with the economic time series. The economic time series as generated by (47) provide for indices of wage inflation and price inflation in each state of the world. If this information is supplemented by the list of participants in the plan and all personal information on participants that is required to determine liabilities of the fund to each participant, then the benefit formulae and actuarial methods of computation can be applied to calculate the level of benefit payments and the actuarial reserve in each state of the world.

Given the development of wage inflation, the career of each employee in each future state of the world and the present reward system, the cost of salaries can be computed for each state of the world.

Thus, in addition to the economic time series, the development of the status of current and future participants in the fund is required to compute the remaining state variables,

i.e. the actuarial reserve, the level of benefit payments and the costs of salaries.

As with the method of generating economic scenarios, our ALM approach is independent of the way in which the development of participants is obtained. Below we briefly discuss a Markov model that has been used in studies for Dutch pension funds as well as for the case study in this monograph. For a more elaborate exposition of this model the reader can be referred to Boender (1994). Markov models in the context of ALM are also presented in Janssen and Mancca (1994).

Modelling the Development of Participants in the Fund

The purpose of modelling the development of participants in the fund is to provide for all relevant personal information on participants that is required to compute liabilities of the fund, and costs of salaries to the company in each future state of the world.

Given the pension-rules and all relevant data on the participants (e.g. civil status, age, gender, salary, earned pension rights, medical status, social status), a Markov model is used to determine the future development of each individual that currently participates in the pension fund. For an employee this implies that each year it is determined whether he remains alive, retires, resigns, gets disabled and/or is promoted to another job category. These transitions are determined by probabilities which depend on characteristics of the individuals such as age, gender and employee-category²¹. Disabled, resigned and retired people in the file are modelled analogously.

Given the situation of each current employee in each future year, the model determines additional promotions and the recruitment of new employees, such that the number of employees in each job category in each future year is in line with prespecified values. The age and gender of new employees are random variables which depend on the categories to which they are assigned.

4.3.4 Generating Future Time Series of Economic and Actuarial Variables

The economic state variables and the corresponding status of the participants in the fund constitute the base information by which the states of the world are described. To construct consistent descriptions of states of the world which include figures that derive from the data that is now available, administrative and actuarial software is used to

²¹ Most of the transition probabilities also depend on the company by which participants are employed and the type of work that they do. This should be taken into account when specifying them.

compute the corresponding actuarial reserve, benefit payments and the cost of salaries for each state of the world. Thus, consistent future time series of states of the world are obtained.

Consistency

To conclude this chapter let us briefly review the concept of consistency of scenarios. The scenario generator that has been described above ensures consistency on various levels. The VAR model ensures consistency over time as well as at points in time between economic state variables.

It is unlikely that arbitrage opportunities will be introduced. It is possible to prevent them from being generated all together by appropriately choosing the last scenario that is generated. One can also check first whether a given set of scenarios incorporates arbitrage opportunities. If it does, then the set of scenarios can be extended by continued random sampling until there are no arbitrage opportunities left. One can also choose to add one well chosen scenario to eliminate all arbitrage opportunities. By applying the scenario generator in combination with one of these procedures, the scenarios are consistent with the widely accepted non arbitrage hypothesis.

The Markov model safeguards consistency of scenarios in terms of the development of participants. Next, the economic data and the data on participants serve as input to actuarial software by means of which the corresponding actuarial data (actuarial reserves and benefit payments) are calculated²².

²² The extent to which it is desirable and feasible to pursue consistency of scenario data in practice may vary. The specification of transition probabilities, for instance, is highly dependent on the pension fund under consideration. In some cases it may very difficult to formulate a longer term personnel policy, let alone transition probabilities from one status to another of future participants to the fund. On the other hand, there may be cases where it is possible to indicate a relationship between the economic environment as reflected by the economic time series and transition probabilities.

Chapter 5

Model Tractability, from Multistage to Two Stage

5.1 Introduction

The ALM model that has been presented in 3.3.2 is a mixed integer problem with $O(S^T)$ variables and equations. For realistic values of S_t and T the problem becomes forbiddingly large to formulate explicitly and to solve within a reasonable amount of time. This is the first of two chapters on reducing the size of the ALM model.

In this chapter we shall describe a heuristic to solve the ALM model. 5.2 presents a scenario structure that can serve to model the uncertain future with a number of states that increases only linearly with T . Given this structure, 5.3 and 5.4 present a procedure to formulate and solve a sequence of two stage problems that serves as an approximation to the ALM model. It defines a procedure that determines decisions backwards, starting at $t=T$ decisions are determined for $t=T-1, T-2, \dots, 1$. The final part of the heuristic consists of a forward procedure which improves the solution from the sequence of two stage models by taking into account information that was not available when the backward procedure was executed: the decisions at preceding points in time and their consequences for the instant of decision making. The entire procedure to compute the approximating solution requires $O(T)$ running time, the constant being bounded by the time required to solve one two stage problem. The solutions to these two stage problems can serve as an approximation to the solution of the multistage ALM model.

Chapter 6 employs a variance reduction technique, importance sampling, to reduce the size of the two stage problems without losing accuracy.

The remaining sections of this chapter present the heuristic in detail. Before turning to this extensive treatment, we shall provide the reader with an informal outline of the procedure. To illustrate this discussion, a small but representative example will be used. Consider the scenario structure, depicted in Figure 8. It is an instance of the scenario structure which was described in paragraph 3.3.1 with $t=0, 1, 2, 3$, $S_1=3$ and $S_t=6$, $t=2, 3$. The set of scenarios depicted in Figure 8 differs from the scenario structure that was pictured earlier in that this set includes scenarios with horizons of different lengths. It has been structured in such a way that there is precisely one path from each state of the world at point in time 1 to the planning horizon, instant 3. All states which are not part of one of these paths, (2,1), (2,4) and (2,5), are terminal states.

This structure has the attractive property that the number of states of the world increases only linearly with the number of decision moments. To appreciate the role of the terminal states consider the situation that state dependent decisions can be made at point in time 1

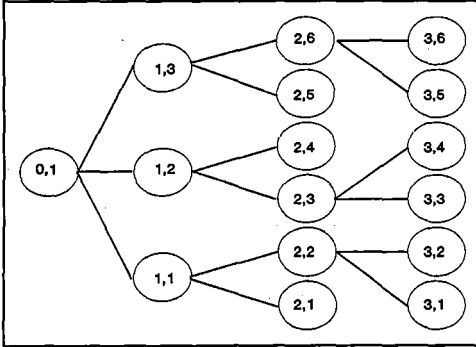


Figure 8 Scenario structure with horizons of different lengths

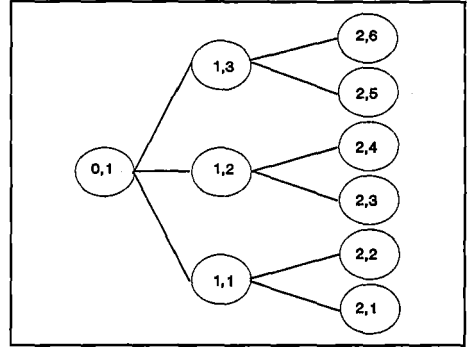


Figure 9 Scenario structure for Twostage2(2)

and that the terminal states (2,1), (2,4) and (2,5) have been removed from the scenarios in Figure 8. Then, there would only be one possible successor for (1,1), (1,2) and (1,3). I.e., at point in time 1, it would be known with certainty which state of the world will emerge at the next point in time. This hindsight could then be exploited, for instance, by allocating all assets to the asset category that is known to give the highest return over period 2. Including states (2,1), (2,4) and (2,5) safeguards the reflection of decision making under uncertainty at point in time 1, it prevents decisions from being driven by hindsight. For sufficiently large S_t , this scenario structure can still serve as an approximation to the underlying distribution since, by the strong law of large numbers, the state variables at given points in time still converge to their underlying continuous distribution.

In order to compute a solution to the ALM problem which is associated with the set of scenarios that is illustrated by Figure 8, we start by determining $Z_{3,s}$, $Y_{2,s}$ and $X_{i2,s} \forall i, s$ by solving the two stage model which is associated with the set of scenarios depicted in Figure 10. This model will be referred to by *Twostage1(3)*. It has been obtained from the original set of scenarios by eliminating all states which do not have descendants at $t=3$, i.e., states (2,1), (2,4) and (2,5) have been removed. Moreover, the first two periods have been aggregated to one period from $t=0$ to $t=2$. This two stage model is solved by means of a branch and bound algorithm. The solution to this model includes the

decisions $Z_{31}, \dots, Z_{36}, Y_{22}, Y_{23}, Y_{26}, X_{i22}, X_{i23}$ and X_{i26} . They are optimal with respect to *Twostage1*(3). If a set of decisions $X_{i01}, Y_{01}, X_{i1s}, Y_{1s}$ can be found, so that the constraints on trading volume and the constraint on advances of regular contributions from period 1 to period 2 are satisfied given Y_{2s} and X_{i2s} , then Y_{2s}, X_{i2s} and Z_{3s} would also be feasible to the multistage problem.

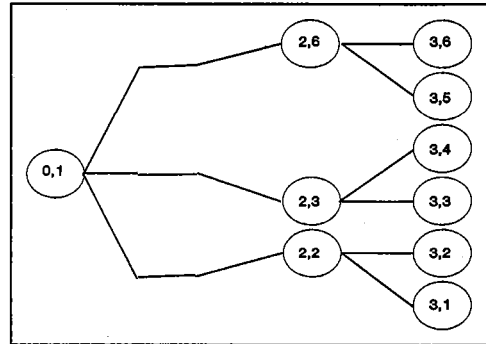


Figure 10 Scenarios for model *Twostage1*(3)

The next step of the procedure consists of formulating *Twostage2*(2) which is associated with the set of scenarios depicted in Figure 9. In addition to constraints that carry over directly from the ALM model, *Twostage2*(2) includes constraints which ensure that a feasible solution $X_{i0s}, Y_{0s}, Z_{1s}, X_{i1s}, Y_{1s}, Z_{2s}$ to *Twostage2*(2), complemented by X_{i2s}, Y_{2s} and Z_{3s} , obtained from the solution to *Twostage1*(3) constitute a feasible solution to the multistage model.

Let us take a closer look at the role that the solution to *Twostage1*(3) plays in the formulation of *Twostage2*(2). The constraints with respect to the state of the pension fund in states (2,2), (2,3) and (2,6) ensure that it is possible to continue an ALM strategy that satisfies the requirements of the multistage ALM model: the solution to *Twostage2*(2) will be derived, subject to feasibility of decisions that were obtained as a solution to *Twostage1*(3). This precludes solutions to *Twostage2*(2) which contain decisions that exploit end effects. I.e., decisions which lead to a reduction of expected costs of funding up to $t=2$ at the cost of ending up in a poor starting situation for period 3 are avoided.

States (2,1), (2,4) and (2,5) have not been included in *Twostage1*(3). Therefore it is not known what values of X and Y in these states would constitute good starting points for the period of time from $t=2$ to $t=3$. In order to specify constraints which prevent end effects from occurring at $t=2$, optimal values for Y and X in (2,1), (2,4) and (2,5) are estimated by means of a metamodel.

This metamodel defines a relationship between the optimal values of X and Y and the available information in states that were not included in the preceding two stage problem.

It is derived from the relationship between optimal values X and Y in states (2,2), (2,3) and (2,6) and the information that is available in these states. Next, estimates for X and Y in states (2,1), (2,4) and (2,5) are obtained as output of the metamodel, using the information that is available in these states as input.

After solving *Twostage2*(2), it may happen that $X_{t=1}$ takes on values that are different from values of $X_{t=1}$ in the optimal solution to *Twostage1*(3). As a consequence, the value of the assets and the surplus at $t=2$ may deviate from the values that have been obtained as part of the solution to *Twostage1*(3) as well. This new information may call for an adjustment of the asset mix and contribution level in states (2,2), (2,3) and (2,6). This is an example of a situation in which decisions that are made in certain stadium of the optimisation procedure call for adjustment of decisions that were made in an earlier stage of the optimisation procedure.

To allow for a mechanism to react to these situations, the heuristic concludes with a forward procedure. Loosely speaking, this forward procedure adjusts decisions at given points in time, given the history by which that point in time was reached, such that the situation is, as much as possible, in line with the one that was created in the backward stage of the optimisation procedure. More specifically: the forward procedure starts at $t=2$. It is checked whether or not the state of the pensionfund is in agreement with the solution to *Twostage1*(3). If yes, then the *Twostage1*(3) decisions remain unchanged. Otherwise, model *Adjust*, which is developed in 5.5, is applied to determine new decisions which are optimal given the changed situation.

The procedure to convert the multistage problem in a series of two stage problems entails several approximations. Firstly, the scenario structure depicted in Figure 1 in which each scenario reflects a path through time from $t=0$ until $t=T$, is replaced by a set of scenarios with horizons of different lengths. The loss of information due to this approximation is negligible: the probability distribution of state variables at given points in time still converges to the underlying continuous distribution by the strong law of large numbers.

The second approximation concerns the solution method. Instead of solving the multistage problem directly, a sequence of two stage models is solved. The solution that has been derived in this way can be proven to be feasible to the multistage problem. However, it is not guaranteed to be optimal since the decisions X_2 and Y_2 which were optimal in

combination with the optimal X_0, X_1, Y_0 and Y_1 to *Twostage1*(3), are not necessarily optimal in combination with the ultimate choice of X_0, X_1, Y_0 and Y_1 . The trade-off between costs which are incurred by decisions at $t=2$ and costs which result from decisions at points in time 0 and 1, does not have to be optimal, because these decisions were not made simultaneously. The decisions at points in time 0 and 1 are optimal, given the decisions that were made earlier at point in time 2. It is not trivial to indicate whether this approximation leads to a systematic bias in the outcomes because the two stage problems are neither relaxations nor special cases of the multistage problem: at intertemporal points in time no recourse decisions can be made, but neither is the policy required to meet actuarial solvency requirements at these points.

The third approximation is due to the metamodel. In states (2,1), (2,4) and (2,5), the constraints that account for end effects are based on estimates of the optimal horizon situations instead of the truly optimal situation as obtained from a preceding two stage solution. In case of inaccurate estimates, these constraints may either be too conservative, which leads to excessively comfortable financial situations of the pension fund at the cost of unnecessarily high contributions in earlier periods, or, if the constraints are not strict enough, the policy determined from the period up to instant t may lead to undesirable starting points for the period from t to T . Notice that the latter type of error can only occur to the extent that the funding level that is required to meet actuarial solvency constraints is less than the optimal funding level.

5.2 Reducing the Number of States of the World

Consider the scenario structure that has been discussed in 3.3.1 and let $S_t = S_2$, $t=2, \dots, T$. This choice of parameter values of the number of nodes in the set of scenarios equals $1 + S_1 + (T-1)S_2$. As can be verified from the ALM model, the number of variables per node is bounded by $3 + 3N$ and the number of equations per node never exceeds $8 + 4N$. Given this set of scenarios, the size of the model is $O(NS_2T)$, it is bounded by a linear function of the number of decision moments, which implies a drastic reduction of the computational complexity of solving the model.

Given the values of S_t , the number of possible states of the world at each point in time has been fixed. Let us now examine the relationship between consecutive states of the world and the probability with which one state succeeds another. Figure 8 depicts an example of the structure that will be used. In general, the structure is defined as follows: at time 0, there is only one state of the world, the one that can be observed, state (0,1).

At the end of the first period, there are S_1 states of the world, which succeed $(0,1)$ with probability p_{1s} . State of the world $(1,s)$, for $s=1, \dots, S_1$, has n_{1s} possible successors, where n_{1s} has been chosen in such a way that the underlying conditional probability distribution of state $(t+1, s')$ given (t,s) is represented sufficiently well, and

$\sum_{s=1}^{S_1} n_{1s} = S_2$. So far, the scenario structure is identical to the one that has been presented

in 3.3.1. For $t > 2$, however, the structure is different. Instead of generating n_{1s} possible successors of $(t,1), (t,2), \dots, (t,S)$, succeeding states are generated for a subset of these states only. The selection of this subset is done in such a way that the history of each state of the world is unique, and that each state of the world at $t=1$ has precisely one descendant at time T . As a consequence, the number of states of the world does not increase exponentially with T any more. Now, it increases linearly with T . Notice that each state in which an ALM decisions can be made has many possible successors. This prevents decisions to be made with hindsight.

It is easy to verify that this structure preserves the requirements on the scenario structure that have been derived in 3.3.1.

The question is now, whether this set of scenarios suffices to approximate the underlying continuous probability distribution of the state variables. Using the strong law of large numbers, it can easily be verified that it does if S_1 and S_2 are chosen sufficiently large. For S_1 and $S_2 \rightarrow \infty$, the sample distribution of the state variables at time t converges stochastically to the underlying distribution, if the mean and variance of this distribution exist (see e.g. Hogg and Craig (1978)). Hence, this scenario structure can serve to describe the uncertain future, provided that S_1 and S_2 are chosen sufficiently large.

5.3 A Sequence of Two Stage Problems to Solve the ALM Model

This paragraph proposes a heuristic to obtain a solution to the ALM model which requires a running time that is bounded by a linear function of T . The solution will be obtained by solving a sequence of two stage problems. Firstly, decisions are obtained for points in time T and $T-1$. Then, working backward, two stage problems are formulated and solved in such a way that their solutions constitute a feasible solution to the ALM model.

Decisions at given points in time affect states of the world at later times. However, the backward procedure does not allow to alter decisions at these times, given decisions at

earlier times. To the extent that states of the world differ from states of the world that were foreseen when the associated decisions were made, the backward procedure may lead to suboptimal solutions. To mitigate this effect, the backward procedure is followed by forward procedure which alters, if desirable, the previously set decisions, based on information from the backward procedure and on the state of the world that has actually emerged.

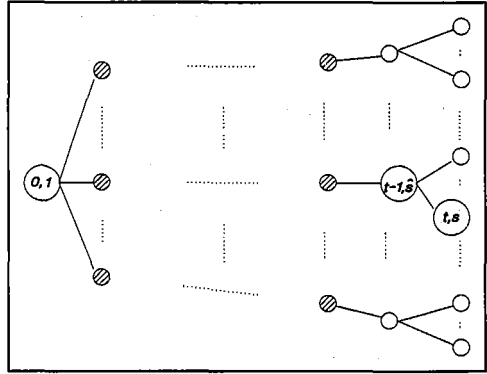


Figure 11 Scenarios for model Twostage(t)

A Sequence of Two Stage Models

In this section model *Twostage(t)* will be developed. It can be viewed as a version of the ALM model which is associated with the set of scenarios depicted in Figure 11. The open circles symbolize states of the world for which decision variables and constraints are included in *Twostage(t)*. The shaded circles symbolize states of the world in which the model does not allow for state dependent decisions. This is the case in all states of the world at points in time $2, \dots, t-2$. In the sense that this model does not admit recourse decisions at these points in time, it cannot be considered to be a relaxation of the ALM model. However, it can't be viewed as a special case of the ALM model either, because solvency requirements and stability constraints are not enforced at intertemporal points in time.

Constraints (78), ..., (83) carry over directly from the ALM model. For their interpretation, the reader can be referred to 3.3.2.

$$A_{ts} = Z_{ts} + \sum_{i=1}^N h_{its} X_{i,t-1,s} \quad s = 1, \dots, S_t \quad (78)$$

$$\sum_{s=1}^{S_t} p[(t,s)|(t-1,\hat{s})] f_{t,s} \leq \Psi_{t-1,\hat{s}}^u \quad s = 1, \dots, S_t \quad (79)$$

$$Y_{\tau s}^l \leq Y_{\tau s} \leq Y_{\tau s}^u \quad \tau = 0, t-1, s = 1, \dots, S_\tau \quad (80)$$

$$y_{\tau s}^l \leq \frac{Y_{\tau s}}{W_{\tau s}} \leq y_{\tau s}^u \quad \tau=0, t-1, s=1, \dots, S_{\tau} \quad (81)$$

$$A_{\tau s} \geq \alpha L_{\tau s} \quad \tau=0, t-1, s=1, \dots, S_{\tau} \quad (82)$$

$$f_{\tau s} \in \{0, 1\} \quad \tau=0, t-1, s=1, \dots, S_{\tau} \quad (83)$$

Aggregation of Period 1, ..., t-2

Cash flows that occur at points in time 1, ..., t-2 are not modelled explicitly any more. They have to be aggregated. Likewise, the return on investments that are made at time 0 has to be compounded for the span of time from 0 to t-1. Benefit payments and contribution payments that are made at time 1, ..., t-2 are reflected as lump sum payments at time t-1. The size of the lump sum payment at time t-1 is computed under the assumption that all intertemporal cash flows are invested in the riskfree asset. I.e., the value of a cashflow to the amount of C at time τ , $1 \leq \tau \leq t-2$, equals

$\exp\left[\sum_{u=\tau-1}^{\tau+1} r_{1us}\right]$ C at time t-1, where r_{1us} denotes the continuous return on the riskfree

asset over period u in the history that leads to state of the world (t-1, s). More generally, a sequence of cash flows C_1, C_2, \dots, C_{t-2} is reflected by a lump sum payment to

the amount of $\sum_{\tau=1}^{t-2} C_{\tau} \exp\left[\sum_{u=\tau-1}^{\tau+1} r_{1us}\right]$. To facilitate notation, the lumpsum payment of a

sequence of cash flows C_1, \dots, C_{t-2} in state (t-1, s) will be denoted by $a_{t-1,s}[C]$. Notice that the aggregation function is linear in the cash flows and that annual riskfree rates of return are scenario dependent. Constraints (84), ..., (88) carry over from the ALM model, after having been adjusted to account for aggregation over time of cash flows and returns.

for $s = 1, \dots, S_{t-1}$

$$A_{t-1,s} = Z_{t-1,s} + \sum_{i=1}^N \exp\left(\sum_{u=1}^{t-1} r_{i u s}\right) X_{i01} \quad (84)$$

$$\frac{Y_{t-1,s}}{W_{t-1,s}} - \frac{Y_{01}}{W_{01}} \leq (t-1) \beta_{t-1,s} \quad (85)$$

$$\sum_{s=1}^{S_{t-1}} p[(t-1, s) | (0, 1)] f_{t-1,s} \leq \psi_{01}^u \quad (86)$$

For $\tau=0, t-1, s=1, \dots, S_\tau$

$$A_{\tau s} + a_{\tau s}[Y - l] = \sum_{i=1}^N X_{i\tau s} \quad (87)$$

$$x_{i\tau s}^l (A_{\tau s} + a_{\tau s}[Y - l]) \leq X_{i\tau s} \leq x_{i\tau s}^u (A_{\tau s} + a_{\tau s}[Y - l]) \quad (88)$$

Reduction of Aggregation Effects

In order to reduce aggregation effects, *Twostage(t)* is brought more in accordance with the ALM model by adding constraints which enforce solvency requirements at a selection of states of the world at points in time $\tau, \tau \in \{1, \dots, t-2\}$. Moreover, the model is extended by allowing for time dependent changes of the level of contributions as a percentage of the cost of wages.

Solvency requirements are enforced in all states of the world in the set of states of the world *Intertemp(t)*. For each possible path from period 1 to period $t-1$, the worst states of the world that emerge along the path are contained in *Intertemp(t)*. The notion worst state of the world is formalised as the states of the world where the cumulative return on at least one asset category, relative to the cumulative growth of the liabilities, is minimal. I.e.,

$$Intertemp(t) = \left\{ (\tau, s) \mid \exists i: \exp\left(\sum_{u=1}^{\tau} r_{i u s}\right) \frac{L_{01}}{L_{\tau s}} \leq \exp\left(\sum_{u=1}^{\tau'} r_{i u s}\right) \frac{L_{01}}{L_{\tau' s}} \forall \tau' \in [1, \dots, t-2] \right\}$$

Thus, given state of the world $(t-1, s)$, there is at least one and there are at most N indices τ such that states of the world (τ, \hat{s}) which are part of the history that led to $(t-1, s)$ are included in *Intertemp(t)*. This extension precludes the choice of asset mixes at time 0 that do well in the long run but would be excessively risky in the short term. This is reflected in *Twostage(t)* by including:

$$\sum_{i=1}^N X_{i01} \exp\left(\sum_{u=\tau}^1 r_{i u s}\right) + a_{\tau s}[Y - l] \geq \alpha L_{\tau, s} \quad \forall (\tau, s) \in Intertemp(t) \quad (89)$$

Secondly, *Twostage(t)* is brought more in accordance with the ALM model by allowing for **time dependent decision rules** with respect to the contribution level. More precisely, in addition to specifying the level of contribution at time 0, the extended model allows for setting an annual change of contribution Δy for period $\tau, \tau=1, \dots, t-2$. The contribution

as a fraction of the cost of wages in state (τ, s) for $\tau = 1, \dots, t-2$ is then equal to

$$\frac{Y_{01}}{W_{01}} + \tau \Delta y. \text{ It follows that } Y_{\tau s} = \left[\frac{Y_{01}}{W_{01}} + \tau \Delta y \right] W_{\tau s}. \text{ Notice that the new decision}$$

variable Δy is state independent. The change in contribution, however, is state dependent, due to the exogenously determined level of the costs of wages. To allow for this extension in the two stage model, while observing the upper bound on hikes in the annual contributions, (85) is replaced by:

$$\Delta y \leq \beta_{t-1,s} \tag{90}$$

$$\frac{Y_{t-1,s}}{W_{t-1,s}} - \frac{Y_{01}}{W_{01}} - (t-2) \Delta y \leq \beta_{t-1,s} \tag{91}$$

Notice that Y in (87) and (88) now refers to the flow of contributions including the annual changes in contribution as defined by Δy .

5.3.1 A Feasible Solution to the ALM Model

Part of the specification of $Twostage(t)$ is motivated by the role that the two stage models play in the heuristic: the solutions to $Twostage$ cannot be considered independently of one another since it should be possible to combine them in order to obtain a solution to the ALM model. Therefore, the focus will now be shifted to the way in which the solutions to consecutive two stage problems constitute a solution to the ALM model. This will impose additional constraints on the solution to each two stage model. $Twostage$ will be extended accordingly.

The first two stage problem that is solved is $Twostage(T)$, i.e., a two stage problem in which decision points 0, $T-1$ and T are modelled explicitly. No recourse decisions can be made at times $2, \dots, T-2$. From the solution to $Twostage(T)$, all variables that do not appear in $Twostage(t)$, for $t < T$ are fixed, i.e., the values obtained for A_{Ts} , Z_{Ts} , $X_{i,T-1,s}$, $Y_{T-1,s}$ and f_{Ts} will not be changed any more in the course of the remaining part of the two stage optimisation procedures. Next, $Twostage(t)$ can be solved for $t = T-1, \dots, 2$ in order to obtain values for A_{ts} , $X_{t-1,s}$, Z_{ts} , f_{ts} and $Y_{t-1,s}$. Thus, a solution to the ALM model can be obtained from the solutions to $T-1$ two stage problems that approximate the ALM model.

Would the solution, obtained as described above, be feasible to the ALM model ? Table 5 shows which values are fixed in which two stage model. Using Table 5, the constraints of $Twostage(t)$ can be classified in two sets: set 1, the set of equations which contain variables that assume their final value simultaneously, i.e. as part of the solution to the same two stage problem, and set 2: the set of equations of which the value of the decision variables is determined by the solution to more than one two stage problem. Constraints (84), (85), (87) and (88) are in set 2²³. All other constraints are specified entirely by variables which assume their final value as part of the solution to one two stage problem, they are in set 1. This distinction is relevant because the solution obtained by subsequently solving $Twostage(t)$ for $t=T, \dots, 2$ and fixing values of decision variables as presented in Table 5, satisfies all constraints in set 1, but constraints in set 2 may be violated.

Table 5. The stage at which decisions are made

Model	Variables of which the value is fixed
$Twostage(T)$	$Y_{T-1}, X_{T-1}, A_T, Z_T, f_T$
:	:
$Twostage(t+1)$	$Y_t, X_t, A_{t+1}, Z_{t+1}, f_{t+1}$
$Twostage(t)$	$Y_{t-1}, X_{t-1}, A_t, Z_t, f_t$
$Twostage(t-1)$	$Y_{t-2}, X_{t-2}, A_{t-1}, Z_{t-1}, f_{t-1}$
:	:
$Twostage(3)$	Y_2, X_2, A_3, Z_3, f_3
$Twostage(2)$	$Y_0, X_0, A_0, Z_1, f_1, Y_1, X_1, A_1, Z_2, f_2$

With respect to feasibility of the solution to the ALM model, set 1 includes all restrictions that define the feasible region of the ALM model, with exception of constraints (35), (36), (38) and (41). These are the equations that reflect stability requirements on the ALM policy. They restrict decisions at given times as a function of the value of decision variables at the preceding decision point. Hence, the procedure that has been described

²³ This does not apply to $Twostage(2)$. Since this is the last two stage problem to be solved, all constraints of this model are in set 1.

above can be applied to solve the ALM model to feasibility if the two stage models are extended with constraints which ensure feasibility with respect to these stability requirements.

All stability requirements can be formulated explicitly as relationships between decision variables at consecutive points in time; they do not restrict the choice of values for decision variables at time $t+k$ in terms of decisions at time t for $|k| > 1$. Hence, it is possible to require solutions to $Twostage(t)$ to satisfy constraints (35), (36), (38) and (41), given the decisions that were obtained as a solution to $Twostage(t+1)$. To enforce this, $Twostage(t)$ can be extended with constraints (92),..., (97):

To ensure feasibility with respect to (35):

$$A_{ts} + Y_{ts}^f - l_{ts} = \sum_{i=1}^N X_{its}^f \quad (92)$$

To ensure feasibility with respect to (36):

$$x_{its}^u (A_{ts} + Y_{ts}^f - l_{ts}) \geq X_{its}^f \quad (93)$$

$$x_{its}^l (A_{ts} + Y_{ts}^f - l_{ts}) \leq X_{its}^f \quad (94)$$

To ensure feasibility with respect to (38):

$$h_{its} X_{i,t-1,s} - C_{its}^u \leq X_{its}^f \quad (95)$$

$$h_{its} X_{i,t-1,s} + D_{its}^u \geq X_{its}^f \quad (96)$$

To ensure feasibility with respect to (41):

$$\frac{Y_{ts}^f}{W_{ts}} - \frac{Y_{t-1,s}}{W_{t-1,s}} \leq \beta_{ts} \quad (97)$$

Although these constraints have to be satisfied to obtain a feasible solution to the ALM model, it is undesirable to impose upper bounds on the value of investments. As long as the minimum levels are attained, one can always do at least as well as the policy that has

been determined by $Twostage(t+1)$. Therefore, (94) and (95) are omitted and (92) is replaced by:

$$A_{ts} \geq l_{ts} - Y_{ts}^f + \sum_{i=1}^N X_{its}^f \quad (98)$$

In the forward procedure that is discussed in 5.5, a decision rule is proposed which safeguards feasibility with respect to (35), (36), (38) and (41). Notice that these additional constraints can be formulated only for states (t,s) for which Y_{ts}^f and X_{its}^f have been determined, i.e. for states (t,s) that were included in $Twostage(t+1)$. For the remaining states of the world horizon conditions will be derived in 5.4. A complete formulation of $Twostage(t)$ is given at page 93.

5.4 A Metamodel to Estimate Optimal Decisions

5.4.1 Introduction

In order to formulate horizon constraints for $Twostage(t)$, a distinction has to be made between nodes (t,s) which were included in $Twostage(t+1)$ and those that were not. The set of nodes that were included in $Twostage(t)$ will be denoted by $Nodes(t)$. Horizon constraints for the nodes in $Nodes(t+1)$ can be derived directly from the solution to $Twostage(t+1)$, as has been explained in 5.3. In this section, the focus is shifted to formulating end constraints which have to be met in states (t,s) that were not included in the preceding two stage problem. This will be done by employing a metamodel that estimates the values for X and Y , that would have been obtained from solving $Twostage(t+1)$ if these states had been included.

5.4.2 Requirements on the Metamodel

Ideally, the metamodel should serve to derive horizon constraints so that decisions that are obtained as solutions to $Twostage(t)$ for $t=T, \dots, 2$ are optimal to the ALM model. In general, this ideal situation will not be attainable. It is possible, however, to estimate minimum values of X_{its} and Y_{ts} which preclude end effects and thus ensure that the pension fund arrives at a good starting position for the period of time from t to T . Moreover, one can provide a measure to value wealth, in excess of the minimum level of asset value that is required to meet solvency requirements in the future. This should be

accomplished without excessive computational effort. After solving $Twostage(t+1)$, the following information is available to serve as input to the metamodel:

- The states of the world at (τ, s) for $\tau=0, t-1, t, s=1, \dots, S_\tau$
- Solutions to $Twostage(t+1)$ for states $(t, s) \in Nodes(t+1)$
- The probability distribution of $R_{t+1, \hat{s}}$ given $R_{t, \hat{s}}$ for $\hat{s}=1, \dots, S_t$.

Notice that the problem to be solved by the metamodel is different from the single period ALM problems that have been discussed in chapter 2. Firstly, here, it is not the intention to arrive at a decision that is optimal with respect to the single period that starts at the moment of decision making. The intention is to estimate decisions that are optimal in the sequence of decisions that are made for $t=0, \dots, T$. Secondly, there is additional information available: the optimal decisions in states of the world in $Nodes(t+1)$.

5.4.3 Mechanisms that Drive the ALM Model

Model ALM is rather complicated to solve. Nevertheless, the forces that drive it are few and they are easy to understand: to minimise expected costs of funding, investment decisions should be made in such a way that the expected return on investments is maximal. However, this goal can be pursued only to the extent that it does not lead to unacceptable risks of underfunding. The risk of underfunding is linked to the probability that the growth of liabilities exceeds the appreciation of the investment portfolio to the extent that a substantial erosion of the surplus occurs.

Given the proportional allocation of assets over the investment classes, the risk of underfunding becomes smaller in accordance with the degree to which the surplus to start with is greater. With difficult times ahead, the multistage character of the ALM model tends to yield solutions which allow for building up a surplus in earlier times that can serve as a buffer under more difficult circumstances. Likewise, rosy perspectives will generally lead to modest surpluses since the costs of maintaining a larger surplus would outweigh the reduction of risk that could be brought about.

The challenge is to formulate a metamodel that reflects these properties while there is no explicit information available on states of the world that succeed $(t, s) \notin Nodes(t+1)$.

5.4.4 A Metamodel to Derive Horizon Constraints

Before turning to the metamodel, let us analyze how the surplus at time $t+1$ depends on

the asset value and asset allocation at time t . The conceptual ALM model that has been developed in 3.2 will serve as a starting point, together with the scenario generator that has been specified by (46).

(19) and (46) imply that the value of the asset portfolio at time $t+1$ is given by

$$A_{t+1} = \sum_{i=1}^N X_{it} \exp[r_{i,t+1}] \quad (99)$$

with $r_{i,t+1} \sim N(E[r_{i,t+1}|r_{it}], \Sigma_r)$

Using a first order Taylor approximation (e.g. Almering et. all (1988)), (99) can be formulated as:

$$A_{t+1} = \sum_{i=1}^N X_{it} (1 + r_{i,t+1} + O(r_{i,t+1}^2)) \quad (100)$$

In the sequel of this section the asset value at time $t+1$ will be modelled as:

$$A_{t+1} = \sum_{i=1}^N (1 + r_{i,t+1}) X_{it} \quad (101)$$

with $r_{i,t+1} \sim N(E[r_{i,t+1}|r_{it}], \Sigma_r)$

This introduces an approximation error of the order of magnitude of $\sum_i X_{it} O(r_{i,t+1}^2)$. It follows that the expected approximation error is bounded by a weighted sum of the variance of r about 0. $r_{i,t+1}$ will typically take on values in the order of magnitude of 0.08. This implies a relative estimation error of the asset value at time $t+1$ of the order of magnitude of 0.6%²⁴.

The distribution of liabilities at time $t+1$ is more difficult to specify. It is a function of price inflation, wage inflation and the development of the characteristics of the partici-

²⁴ Actually, one can obtain better Taylor approximations by approximating the exponential powers about their expected value. The remaining analyses remain unchanged when this is done.

pants. This function is defined by the benefit formulae and the actuarial standards. Given a specific pension plan, the specification can be extracted from the benefit formulae.

Here, it will be assumed that liabilities can be divided in four components: a constant, a component that is indexed with price inflation, a component that is indexed with wage inflation and, finally, a component that depends on a number of unspecified factors, which are assumed not to be state dependent. These assumptions are reflected by (102):

$$L_{t+1} = \eta_{0t} + \eta_{1t} \exp[g_{t+1}] L_t + \eta_{2t} \exp[w_{t+1}] L_t + \eta_{3t} L_t \quad (102)$$

with $\eta_{0t}, \eta_{1t}, \eta_{2t} \geq 0$

As with the asset value, the value of liabilities at time $t+1$ will be approximated by using a first order Taylor approximation of $\exp[g_{t+1}]$ and $\exp[w_{t+1}]$:

$$L_{t+1} = \eta_{0t} + \eta_{1t}(1+g_{t+1})L_t + \eta_{2t}(1+w_{t+1})L_t + \eta_{3t}L_t \quad (103)$$

The coefficients η specify the relative sizes of the components which constitute the liabilities at time t . They can be estimated from observations in pairs of states (t, δ) , $(t+1, s)$ in $Nodes(t+1)$. Estimating them can thus be done by solving for η in regression model (104). Notice that one may expect to obtain a very good fit because (104) virtually reflects the administrative definition equation that is implicitly given by the actuarial rules that are applied to determine the level of liabilities.

$$L_{t+1,s} = \eta_{0t} + \eta_{1t}(1+g_{t+1,s})L_{t\delta} + \eta_{2t}(1+w_{t+1,s})L_{t\delta} + \eta_{3t}L_{t\delta} \quad (104)$$

Given the distribution of the asset value and the liabilities, the surplus at time $t+1$ can be approximated by:

$$B_{t+1} = \sum_{i=1}^N (1+r_{i,t+1})X_{it} - \eta_{0t} - (\eta_{1t}(1+g_{t+1}) + \eta_{2t}(1+w_{t+1}) + \eta_{3t})L_t \quad (105)$$

Since r , g and w are normally distributed, the level of the surplus at time $t+1$ is a linear combination of normally distributed random variables. This implies that B_{t+1} is

distributed normally and that the mean and variance of this distribution are determined by the choice of X_t . This brings us back in the domain of mean variance type ALM models.

The Minimally Required Level of Asset Value

First, let us determine the minimum level of the value of assets that is required at the beginning of period t to meet solvency constraints at time $t + 1$ with probability ψ_t^u . This is done by utilizing a variant of the chance constrained programming model that has been discussed in chapter 2:

Min Assets(t, δ)

$$\text{Min } \sum_{i=1}^N X_{it\delta} \tag{106}$$

s.t.

$$\sum_{i=1}^N X_{it\delta} \geq \alpha L_{t\delta} \tag{107}$$

$$E[B_{t+1,s}] = \sum_{i=1}^N (1 + E[r_i|(t, \delta)]) X_{it\delta} - \eta_{0t} - \tag{108}$$

$$(\eta_{3t} + \eta_{1t}(1 + E[g_{t+1,s}|(t, \delta)]) + \eta_{2t}(1 + E[w_{t+1,s}|(t, \delta)])) L_{t\delta}$$

$$\sigma^2[A_{t+1,s}] = \sum_{i,j=1}^N X_{it\delta} \sigma[r_i, r_j] X_{jt\delta} \tag{109}$$

$$\sigma[A_{t+1,s}, L_{t+1,s}] = \sum_{i=1}^N X_{it\delta} \sigma[r_i, r(L_{t\delta})_{t\delta}] L_{t\delta} \tag{110}$$

$$\sigma^2[B_{t+1,s}] = \sigma^2[A_{t+1,s}] + \sigma^2[L_{t+1,s}] - 2\sigma[A_{t+1,s}, L_{t+1,s}] \tag{111}$$

$$E[B_{t+1,s}]^2 \geq (\varphi^{-1}(\psi_{it\delta}^u))^2 \sigma^2[B_{t+1,s}] \tag{112}$$

$$x_{it\delta}^l (\sum_{i=1}^N X_{it\delta}) \leq X_{it\delta} \leq x_{it\delta}^u (\sum_{i=1}^N X_{it\delta}) \tag{113}$$

$$E[B_{t+1,s}] \geq 0 \tag{114}$$

MinAssets(t, δ) computes the minimal asset value that is required in state (t, δ) , in order to satisfy the probabilistic solvency constraint on a minimum surplus at the end of period t . (107) ensures that the funding requirement at the beginning of period t is met as well.

After solving $MinAssets(t, s)$ for $s = 1, \dots, S_t$, a mean variance efficient estimate of the minimally required asset level and the associated asset mix have been obtained for all terminal states of $Twostage(t)$. By using X_{its}^m , the solution to $MinAssets(t, s)$, as a lower bound on X_{its} in $Twostage(t)$, any feasible solution to $Twostage(t)$ constitutes a feasible starting point for period t . Therefore, the following constraints are included for $(t, s) \notin Nodes(t+1)$:

$$A_{ts} + (1 + \beta_{t-1,s}) \frac{Y_{t-1,\hat{s}}}{W_{t-1,\hat{s}}} y_{ts} \geq \sum_{i=1}^N X_{its}^m \quad (115)$$

$$h_{its} X_{i,t-1,\hat{s}} + D_{its}^u \geq X_{its}^m \quad i = 1, \dots, N \quad (116)$$

$$x_{its}^u (A_{ts} + (1 + \beta_{t-1,s}) \frac{Y_{t-1,\hat{s}}}{W_{t-1,\hat{s}}} W_{ts} - l_{ts}) \geq X_{its}^m \quad i = 1, \dots, N \quad (117)$$

Clearly, these constraints are not overly restrictive. It is conceivable, however, that an optimal multistage solution would call for asset levels in excess of X_{its}^m . The next section proposes a method to determine whether, and if so, by how much asset levels should exceed X_{its}^m . The method is based upon mean variance theory and exploits the information that is available from the solution to $Twostage(t+1)$.

The Optimal Asset Level and Asset mix

For states $(t, s) \in Nodes(t+1)$, X_{its}^f is available, as well as X_{its}^m . Let $X_{its}^+ = X_{its}^f - X_{its}^m$ and $A_{its}^+ = \sum_{i=1}^N X_{its}^+$. Then, A_{its}^+ can be interpreted as the amount that, under the optimal policy, should be invested in excess of the minimally required amount. Given the probability distribution of the investment returns in state (t, \hat{s}) , the standard deviation of the return on the additional investments over period t in state $(t+1, s)$ equals:

$$\sigma^2[r(A_{its}^+)_{ts}] = \sum_{i,j=1}^N \frac{X_{its}^+}{A_{its}^+} \sigma[r_i, r_j] \frac{X_{jts}^+}{A_{its}^+} \quad (118)$$

Assuming that the optimal standard deviation depends linearly²⁵ on the additional level of assets, the following regression model can be specified:

$$\sigma[r(A_{ts}^+)] = \theta_0 + \theta_1 A_{ts}^+ + \epsilon_{ts} \quad (119)$$

(119) is the estimated relationship between the optimal standard deviation of the return on the asset value in addition to the minimally required level and the level of additional assets.

For nodes $(t, s) \notin \text{Nodes}(t+1)$, the capital market line, as is well known from mean variance investment theory, defines a linear relationship between standard deviation and expected return of mean variance efficient portfolios:

$$E[r(A_{ts}^+)] = k_0 + k_1 \sigma[r(A_{ts}^+)] \quad (120)$$

which implies:

$$\sigma[r(A_{ts}^+)] = \frac{E[r(A_{ts}^+)] - k_0}{k_1} \quad (121)$$

Substituting (119) in (121) gives

$$E[r(A_{ts}^+)] = k_0 + k_1(\theta_0 + \theta_1 A_{ts}^+) \quad (122)$$

which relates the expected return under the optimal asset allocation of an additional unit of wealth to the level of additional wealth. For instance, let the additional wealth be equal to A_{ts}^+ . Then, the expected additional revenues from the asset portfolio over period t are equal to $(k_0 + k_1(\theta_0 + \theta_1 A_{ts}^+))A_{ts}^+$. Taking the first derivative to A_{ts}^+ it can be shown that one unit of additional wealth in state (t, s) is equivalent to $k_0 + k_1\theta_0 + 2k_1\theta_1 A_{ts}^+$ units of additional expected wealth in $(t+1, s)$. Notice that one unit of additional investments at time t can lead to a more than proportional increase in the expected level of assets at

²⁵ In order to obtain a more accurate reflection of the relationship between expected return and excess investments, a piecewise linear relation can be estimated in an analogous way. This does not affect the remaining analyses in a substantial way.

time $t+1$. This is caused by the fact that a higher surplus allows for a riskier asset mix which comes with a higher expected return for the entire asset portfolio.

Now the value of an additional unit to invest in state (t, \hat{s}) has been established, it is possible to incorporate the trade-off between costs to generate the additional unit to invest and the benefits from the additional wealth. Using the discount factors from the objective function of the ALM model, the present value of one unit of additional wealth in state $(t+1, s)$ equals $\gamma_{t+1,s}(k_0 + k_1\theta_0 + 2k_1\theta_1 A_{ts}^*)$.

These results will be used to quantify the trade-off between additional costs of funding and the benefits from a higher surplus at the outset of period t in states of the world $(t, s) \notin \text{Nodes}(t+1)$. This will be discussed in 5.4.5.

5.4.5 Selecting an Objective Function for $\text{Twostage}(t)$

A natural way to choose an objective function for $\text{Twostage}(t)$ would be to take the objective function of the ALM model and adjust it for the aggregation over the period from decision point 1 to decision point $t-1$. However, in that case the objective function would only reflect the cost of funding of up to time t . Given the analysis above, the objective function can be extended by a component that reflects the value of finishing at time t in a position that is better than the minimally required position.

In order to incorporate the trade-off in $\text{Twostage}(t)$, credit should be given for a larger than necessary asset value. This has been done by specifying the following objective for $\text{Twostage}(t)$:

Minimise $V = A_{01} + Y_{01} +$

$$\sum_{s=1}^{S_{t-1}} p_{t-1,s} \gamma_{t-1,s} (a_{t-1,s} [Y_{01} + \Delta y W] + Y_{t-1,s}) + \quad (123)$$

$$\sum_{s=1}^{S_t} p_{ts} \gamma_{ts} (-2k_1 \theta_1 \sum_{i=1}^N (X_{its} - X_{its}^m) + \lambda Z_{ts})$$

The other side of the coin, the costs, were already included since the objective is to minimise expected costs of funding.

5.5 A Forward Procedure

The forward procedure that is described here serves two purposes:

- adjusting the solutions to $Twostage(T), \dots, Twostage(3)$ in such a way that they constitute a feasible solution to the ALM model,
- improving the solutions by altering the decisions made in state (t, s) in such a way that decisions at earlier times and their consequences are taken into account. Notice that this information was not available when the associated two stage models were solved.

The idea behind the forward procedure is, that given state of the world (t, s) , the decisions X_{its}^f, Y_{is}^f are part of a good solution to the ALM model in view of the span of time from t to T . However, when the values of X_{its}^f and Y_{is}^f were fixed, the decisions at preceding points in time were not yet known. Given this new information it may be possible to do better than just choosing X_{its}^f and Y_{is}^f . For instance, if the surplus in state (t, s) turns out to be substantially lower than expected, then it may be optimal to choose a less risky asset mix. Vice versa, if the surplus is unexpectedly high, it may be optimal to cut contributions or to select a riskier asset mix. It may also happen that X_{its}^f and Y_{is}^f are infeasible to the ALM model, given the new information. Model $Adjust(t, s)$ minimises deviations from the ALM policy that has been specified by $X_{its}^f, Y_{is}^f, A_{is}^f$ and Z_{is}^f , subject to being feasible to the ALM model.

Given state $(t, s) \in Nodes(t+1)$, $2 \leq t \leq T-1$, (37) and (39) imply

$$Z_{is} = \max(0, \alpha L_{is} - \sum_{i=1}^N h_{its} X_{i,t-1,s}), \text{ which, in conjunction with (37) fixes } A_{is}:$$

$$A_{is} = Z_{is} + \sum_{i=1}^N h_{its} X_{i,t-1,s}. \text{ In order to satisfy the constraints of the ALM model with}$$

respect to decisions to be made in state (t, s) , constraints (33), ..., (36), (38) and (41) have to be met. Moreover, in order to be able to pursue the ALM policy in succeeding states at instant $t+1$, (37), ..., (43) should be satisfied. This leads to model $Adjust$ at page 90.

If there is no solution to $Adjust(t, s)$ which satisfies (136), ..., (139), then it cannot be

Adjust(t, s)

$$\text{Minimise } V = A_{ts} + Y_{ts} + \sum_{s':\hat{s}'=s} p_{t+1,s'} \gamma_{t+1,s'} (-2k_1 \theta_1 \sum_{i=1}^N (X_{i,t+1,s'} - X_{i,t+1,s'}^m) + \lambda Z_{t+1,s'}) \quad (124)$$

s.t.

$$Y_{ts}^l \leq Y_{ts} \leq Y_{ts}^u \quad (125)$$

$$y_{ts}^l \leq \frac{Y_{ts}}{W_{ts}} \leq y_{ts}^u \quad (126)$$

$$A_{ts}^a + Y_{ts} - l_{ts} = \sum_{i=1}^N X_{its} \quad (127)$$

$$x_{its}^l (A_{ts}^a + Y_{ts} - l_{ts}) \leq X_{its} \leq x_{its}^u (A_{ts}^a + Y_{ts} - l_{ts}) \quad (128)$$

$$h_{its} X_{i,t-1,\hat{s}}^a - C_{its}^u \leq X_{its} \leq h_{its} X_{i,t-1,\hat{s}}^a + D_{its}^u \quad (129)$$

$$\frac{Y_{ts}}{W_{ts}} - \frac{Y_{t-1,\hat{s}}^a}{W_{t-1,\hat{s}}} \leq \beta_{ts} \quad (130)$$

for $s':\hat{s}' = s$

$$A_{t+1,s'} = Z_{t+1,s'} + \sum_{i=1}^N h_{i,t+1,s'} X_{its} \quad (131)$$

$$A_{t+1,s'} \geq \alpha L_{t+1,s'} \quad (132)$$

$$Z_{t+1,s'} \leq f_{t+1,s'} M_{t+1,s'} \quad (133)$$

$$\sum_{s'=1}^{s_{t+1}} p[(t+1,s')|(t,s)] f_{t+1,s'} \leq \Psi_{ts}^u \quad (134)$$

$$f_{t+1,s'} \in \{0,1\} \quad (135)$$

for $s' : \hat{s}' = s, (t, s') \in \text{Nodes}(t+1)$

$$\frac{Y_{t+1,s'}^f}{W_{t+1,s'}} - \frac{Y_{ts}}{W_{ts}} \leq \beta_{t+1,s'} \quad (136)$$

$$X_{i,t+1,s'}^f \leq h_{i,t+1,s'} X_{its} + D_{i,t+1,s'}^u \quad (137)$$

for $s' : \hat{s}' = s, (t, s') \notin \text{Nodes}(t+1)$

$$\frac{Y_{t+1,s'}^m}{W_{t+1,s'}} - \frac{Y_{ts}}{W_{ts}} \leq \beta_{t+1,s'} \quad (138)$$

$$X_{i,t+1,s'}^m \leq h_{i,t+1,s'} X_{its} + D_{i,t+1,s'}^u \quad (139)$$

guaranteed that the policy that has been obtained from the backward procedure can be continued at the end of period t . In that case, these constraints are removed. Instead, the objective function is changed so as to minimise the violation of these constraints in order to safeguard the ability to continue the desired policy to the maximum extent.

The decisions to be made at points in time 0 and 1 are determined by *Twostage(2)*. It is easy to verify that these decisions do not violate any constraints of the ALM model. Given these decisions, the forward procedure proceeds as follows:

for $t = 2, \dots, T-1, (t, s) \in \text{Nodes}(t+1)$ do

- Compute A_{ts} and set Z_{ts} at its minimum, which is uniquely determined by constraints (37), (39) and (40).

- Determine the values $X_{its}^a, Y_{ts}^a, A_{t+1,s'}^a, Z_{t+1,s'}^a,$ and $f_{t+1,s'}^a$ for $s' : \hat{s}' = s$ by solving *Adjust(t, s)*.

endfor

The proposed ALM policy is given by $X^a, Y^a, A^a, Z^a,$ and f^a .

5.6 Summary

In this chapter the following heuristic has been developed to approximate the multistage ALM model:

Backward Procedure

Step 0. Solve $Twostage(T)$

$$\text{Set } X_{i,T-1,s}^f = X_{iT-1,s}, Y_{T-1,s}^f = Y_{T-1,s}, Z_{Ts}^f = Z_{Ts}, f_{Ts}^f = f_{Ts}, A_{Ts}^f = A_{Ts}.$$

Step 1. For $t = T-1, \dots, 2$ do

Solve $Twostage(t)$

$$\text{Set } X_{i,t-1,s}^f = X_{it-1,s}, Y_{t-1,s}^f = Y_{t-1,s}, Z_{ts}^f = Z_{ts}, f_{ts}^f = f_{ts}, A_{ts}^f = A_{ts}$$

End for

Step 2. Set $X_{i01}^f = X_{i01}, Y_{01}^f = Y_{01}.$

Forward Procedure

Step 3. for $t = 2, \dots, T-1, (t, s) \in Nodes(t+1)$ do

- Compute A_{ts} and set Z_{ts} at its minimum, which is uniquely determined by constraints (37), (39) and (40).

- Determine the values $X_{its}^a, Y_{ts}^a, A_{t+1,s'}^a, Z_{t+1,s'}^a,$ and $f_{t+1,s'}^a$ for $s' : \hat{s}' = s$ by solving $Adjust(t, s).$

The proposed ALM policy is given by $X^a, Y^a, A^a, Z^a,$ and $f^a.$

$Twostage(t)$ refers to the model at page 93.

Twostage(t)

Minimise $V = A_{01} + Y_{01} +$

$$\sum_{s=1}^{S_{t-1}} p_{t-1,s} \gamma_{t-1,s} (a_{t-1,s} [Y_{01} + \Delta y W] + Y_{t-1,s}) + \tag{140}$$

$$\sum_{s=1}^{S_t} p_{ts} \gamma_{ts} (-2k_1 \theta_1 \sum_{i=1}^N (X_{its} - X_{its}^m) + \lambda Z_{ts})$$

s.t.

for $s = 1, \dots, S_t$

$$A_{ts} = Z_{ts} + \sum_{i=1}^N h_{its} X_{i,t-1,s} \tag{141}$$

$$\sum_{s=1}^{S_t} p[(t,s)|(t-1,\hat{s})] f_{t,s} \leq \Psi_{t-1,\hat{s}}^u \tag{142}$$

For $\tau = 0, t-1, s = 1, \dots, S_\tau$

$$Y_{\tau s}^l \leq Y_{\tau s} \leq Y_{\tau s}^u \tag{143}$$

$$y_{\tau s}^l \leq \frac{Y_{\tau s}}{W_{\tau s}} \leq y_{\tau s}^u \tag{144}$$

$$A_{\tau s} \geq \alpha L_{\tau s} \tag{145}$$

$$f_{\tau s} \in \{0, 1\} \tag{146}$$

for $s = 1, \dots, S_{t-1}$

$$A_{t-1,s} = Z_{t-1,s} + \sum_{i=1}^N \exp\left(\sum_{u=1}^{t-1} r_{iu\hat{s}}\right) X_{i01} \tag{147}$$

$$\frac{Y_{t-1,s}}{W_{t-1,s}} - \frac{Y_{01}}{W_{01}} \leq (t-1) \beta_{t-1,s} \tag{148}$$

$$\sum_{s=1}^{S_{t-1}} p[(t-1,s)|(0,1)] f_{t-1,s} \leq \Psi_{01}^u \tag{149}$$

For $\tau=0, t-1, s=1, \dots, S_\tau$

$$A_{\tau s} + a_{\tau s}[Y - l] = \sum_{i=1}^N X_{i\tau s} \quad (150)$$

$$x_{i\tau s}^l(A_{\tau s} + a_{\tau s}[Y - l]) \leq X_{i\tau s} \leq x_{i\tau s}^u(A_{\tau s} + a_{\tau s}[Y - l]) \quad (151)$$

for $\hat{s} = 1, \dots, S_{t-1}$

$$\Delta y \leq \beta_{t-1, \hat{s}} \quad (152)$$

$$\frac{Y_{t-1, \hat{s}}}{W_{t-1, \hat{s}}} - \frac{Y_{01}}{W_{01}} - (t-2)\Delta y \leq \beta_{t-1, \hat{s}} \quad (153)$$

for $s = 1, \dots, S_t, (t, s) \in \text{Nodes}(t+1)$

$$A_{ts} \geq l_{ts} - Y_{ts}^f + \sum_{i=1}^N X_{its}^f \quad (154)$$

$$x_{its}^u(A_{ts} + Y_{ts}^f - l_{ts}) \geq X_{its}^f \quad (155)$$

$$h_{its} X_{i, t-1, \hat{s}} + D_{its}^u \geq X_{its}^f \quad (156)$$

for $s = 1, \dots, S_t, (t, s) \notin \text{Nodes}(t+1)$

$$A_{ts} + (1 + \beta_{t-1, \hat{s}}) \frac{Y_{t-1, \hat{s}}}{W_{t-1, \hat{s}}} y_{ts} \geq \sum_{i=1}^N X_{its}^m \quad (157)$$

$$X_{i, t-1, \hat{s}} e^{r_{its}} + D_{its}^u \geq X_{its}^m \quad (158)$$

$$x_{its}^u(A_{ts} + (1 + \beta_{t-1, \hat{s}}) \frac{Y_{t-1, \hat{s}}}{W_{t-1, \hat{s}}} W_{ts} - l_{ts}) \geq X_{its}^m \quad (159)$$

$$\sum_{i=1}^N X_{i01} \exp\left(\sum_{u=\tau}^1 r_{iu\hat{s}}\right) + a_{\tau s}[Y - l] \geq \alpha L_{\tau, s} \quad \forall (\tau, s) \in \text{Intertemp}(t) \quad (160)$$

$$\frac{Y_{ts}^f}{W_{ts}} - \frac{Y_{t-1, \hat{s}}}{W_{t-1, \hat{s}}} \leq \beta_{ts} \quad (161)$$

Chapter 6

Model Tractability, Variance Reduction

6.1 Introduction

The approach to the ALM problem that is proposed in this monograph can be characterised as a hybrid simulation and optimisation procedure: first, the development of the environment is being simulated, then, an optimisation procedure is employed in order to determine optimal decisions. The exogenous environment, the contingent decisions to be made and the dynamics by which the value of assets and liabilities carries over from one point in time to the next constitute a system that simulates the development of the pension fund under a variety of future circumstances, assuming that the optimal policy will be implemented.

The fundamental role of the concept of simulation in our approach is evident. There seems to be no consensus in the literature on a unique and useful definition of simulation. In this thesis we shall use one of definitions given in Kleijnen (1971). He defines **simulation** as "experimenting with an (abstract) model over time, this experimenting involving the sampling of values of stochastic variables from their distribution. Because random numbers are used, this type of simulation is sometimes denoted as **Monte Carlo simulation**". The experiments are usually aimed at determining the value of one or more unknown variables. These variables are referred to as the **response variables** of the simulation system. Due to the stochasticity in the simulation process, the value of the response variables is dependent on the sample of values of the stochastic variables that are involved. Thus, different samples may lead to different values of the response variables. As a consequence of the use of random sampling, the outcome of the simulation is random itself.

Let us assume that the simulation system has been designed in such a way that the values of response variables yield unbiased estimates of the true values, i.e., when the sample size goes to infinity, the estimated values of the response variables converge to the true values of the response variables. The inaccuracy of the estimates can then be quantified by the variance of the estimator. One would thus be interested in **unbiased estimates with a minimal variance**. The variance can be reduced by increasing the sample size, at the cost of the corresponding computational effort. One can also obtain more reliable estimates without increasing the sample size by the employment of variance reduction techniques. For an extensive treatment of the theory of simulation, the reader can be referred to Naylor et al (1967), Kleijnen (1974,1975) and Bratley, Fox and Schrage (1987).

In the case of the ALM model, the size of the two stage models is determined by the number of states of the world that is included. The fewer states of the world the smaller the computational effort to solve the models. Thus, the fewer the better. On the other hand, the number of states of the world should be sufficiently large to represent the underlying continuous probability distribution. In this chapter, a variance reduction technique, importance sampling, will be employed to reduce the number of states of the world that is required to obtain a sufficiently accurate estimate of the optimal objective function value.

The next section presents a general exposition of the main idea behind importance sampling, the way in which it can be applied to reduce the size of stochastic linear programmes and a characterisation of optimal importance sampling distributions.

6.2 Importance Sampling

6.2.1 The Potential Merits of Importance Sampling

The basic idea of importance sampling is to replace the original sampling process by another one. This distortion is corrected by weighing the observations from the new sampling process, so that the average of the weighted observations is an unbiased estimator of the mean of the original process. When the new sampling process is designed well, a substantial reduction of the variance of the estimator can be achieved.

Let ξ be a discrete random variable with probability space $\mathcal{E} = \{\xi_1, \xi_2, \dots, \xi_{S^p}\}$ and density function $p(\xi)$, $p(\xi_s) = p_s$ for $s = 1, \dots, S^p$, $p(\xi) = 0$ elsewhere. Suppose that one would like to estimate the mean of the random variable $c(\xi)$:

$$E_p[c(\xi)] = \sum_{s=1}^{S^p} p_s c(\xi_s) \quad (162)$$

To obtain an estimate of $E_p[c(\xi)]$, one can generate a random sample of size S from p and calculate the sample mean of the observations:

$$c_S^p = \frac{1}{S} \sum_{s=1}^S c(\xi_s) \quad (163)$$

It is well known that c_s^p is an unbiased estimator of $E_p[c(\xi)]$, given a sample $\xi_1, \xi_2, \dots, \xi_s$ from p .

Let $q(\xi)$ be a density function with $q(\xi) > 0 \Leftrightarrow p(\xi) > 0$. Then, $q(\xi)$ can be written as $q(\xi_s) = q_s$ for $s = 1, \dots, S^p$, $q(\xi) = 0$ elsewhere. We shall refer to q as an **importance sampling distribution**. Given p and q ,

$$\begin{aligned} E_p[c(\xi)] &= \sum_{s=1}^{S^p} p_s c(\xi_s) \\ &= \sum_{s=1}^{S^p} q_s \left(\frac{p_s c(\xi_s)}{q_s} \right) \\ &= E_q \left(\frac{p_s c(\xi)}{q_s} \right) \end{aligned} \tag{164}$$

It follows that $1/S \sum_{s=1}^S c(\xi_s) p_s / q_s$, the mean of sample $c(\xi_1) p_1 / q_1, \dots, c(\xi_s) p_s / q_s$ from probability density function q , is also an unbiased estimate of $E_p[c(\xi)]$.

To demonstrate the advantage of importance sampling, suppose that q_s is chosen as follows:

$$q_s = \frac{p_s c(\xi_s)}{E_p[c(\xi)]} \tag{165}$$

Let c_s^q denote the average of a sample of size S from density q . Then, the variance of c_s^q is given by (166).

I.e., even with a sample size of 1, the variance of the estimator has been reduced to zero. Of course, if one is able to choose q_s as specified in (165), the value of $E_p[c(\xi)]$ would already be known and one would not have to estimate it any more. Nevertheless, this analysis provides insight in the potential merits of employing importance sampling. Notice that the reduction of variance is independent of the sample size. Moreover, it is useful as a characterisation of the optimal importance sampling distributions: the maximal variance

$$\begin{aligned}
\sigma^2[c_s^q] &= \frac{1}{S} \sum_{s=1}^S \left[\frac{p_s c(\xi_s)}{q_s} - E_q \left[\frac{p(\xi) c(\xi)}{q(\xi)} \right] \right]^2 \\
&= \frac{1}{S} \sum_{s=1}^S \left[\frac{p_s c(\xi_s)}{q_s} - E_p [c(\xi)] \right]^2 \\
&= \frac{1}{S} \sum_{s=1}^S \left[\frac{p_s c(\xi_s)}{p_s c(\xi_s)} E_p [c(\xi)] - E_p [c(\xi)] \right]^2 \\
&= 0
\end{aligned} \tag{166}$$

reduction is achieved if $p_s c(\xi_s)/q_s = E_p[c(\xi)] \forall s$. For a continuous probability space analogous results can be obtained. Appendix C contains an example that illustrates that the variance of the estimator can be reduced to zero if the optimal importance sampling distribution is used. It also contains an example which shows that it is possible to obtain a substantial variance reduction when the importance sampling distribution is not chosen optimally.

When q is chosen well, the result is an estimator with a smaller variance than obtained with the original distribution. When q is chosen badly, however, the variance of the estimator may increase. The effect of importance sampling will be greater in accordance with the degree to which the importance sampling distribution differs more from the original distribution and in accordance with the degree to which the range of values that $c(\xi)$ can assume is greater. Glynn and Iglehart (1989) and Kleijnen (1972) provide methods for specifying importance sampling distributions for some specific types of sampling distributions and response variables. In 6.3 we propose an importance sampling that distribution is suitable for use within the framework of our ALM approach.

6.2.2 Importance Sampling in Stochastic Linear Programming

A **stochastic linear program** is a linear program of which one or more of the coefficients are random variables. The ALM model is more complicated because it is a mixed integer programme. But since the issues that are of interest to stochastic linear programming are relevant to the ALM model as well, we shall shortly discuss the use of importance sampling in stochastic linear programming. The interested reader can be referred to Infanger (1992) for a more detailed exposition. Entriken and Infanger (1990) contains an application of importance sampling in stochastic linear programming, under the assumption that the random input variables are distributed independently.

The aspects that are of importance to the employment of **importance sampling in simulation** play a role in stochastic linear programming as well. Using importance sampling in a simulation experiment is usually aimed at variance reduction of the response variables. The specification of an appropriate importance sampling distribution requires considerable knowledge of the relationship between stochastic input variables and response variables (respectively ξ and $c_s^p(\xi)$ in 6.2.1). In case of a simulation system, the relationship between ξ and $c(\xi)$ may be a complex one (otherwise, one might not have resorted to simulation in the first place) which is only given implicitly. Moreover, one may be interested in several response variables, whereas it may be difficult to specify an importance sampling distribution that reduces the variance of all response variables; in fact, it is possible that an importance sampling distribution which is well suited to reduce the variance of one response variable increases the variance of another.

When using **importance sampling in stochastic programming**, there is an additional issue of interest: the response variable will usually be the optimal value of a decision variable or the optimal objective function value. As a consequence it is usually very difficult to specify an explicit analytical relationship that approximates the true relationship between stochastic input variables and the response variables well. For example, consider *SLP direct*:

SLP direct

$$\underset{x}{\text{Minimise}} \quad \sum_{s=1}^{S^*} p_s \sum_{i=1}^N c_{si} x_i \quad (167)$$

s.t.

$$\sum_{i=1}^N \xi_{si} x_i \leq \xi_{s0} \quad s = 1, \dots, S^* \quad (168)$$

It minimises the expected costs over scenarios $1, \dots, S^*$. The minimisation takes place, subject to satisfying a set of constraints that reflect requirements on the decision x in each scenario that can occur. Given this set of scenarios, *SLP direct* can be solved by means of a standard LP solver. The expected computational effort that is required to solve it depends on the size of the problem, which in turn is a function of S^* , the number of scenarios that is taken into account. Let S^* be the smallest number of scenarios that has to be sampled from p in order to represent the distribution of ξ sufficiently well to obtain an acceptably accurate estimate of the optimal objective function value of

SLP direct.

Given q and an importance sample of size S , *SLP importance* can be formulated as:

SLP importance

$$\text{Minimise } x \quad \sum_{s=1}^S q_s \sum_{i=1}^N \frac{p_s}{q_s} c_{is} x_i \quad (169)$$

s.t.

$$\sum_{i=1}^N \xi_{si} x_i \leq \xi_{s0} \quad s = 1, \dots, S \quad (170)$$

If S and S^* are chosen sufficiently large, then, for any choice of x , the objective function value of both programmes will be identical because they converge to the mean of the cost function when $S, S^* \rightarrow \infty$. As a result, the optimal solutions to both programmes are also identical for sufficiently large samples. However, if the importance sampling distribution has been chosen well, then for any sample size, the variance of the optimal objective function of *SLP importance* will be smaller than that of *SLP direct*. To obtain the accuracy that would be achieved by solving *SLP direct* with a sample size of S^* , one could also solve *SLP importance* with a sample size $S < S^*$ which would reduce the size of the LP to solve and thereby the computational effort to obtain the desired solution. Nakayama (1989) claims a variance reduction of 1:20000 using importance sampling versus crude Monte Carlo sampling. In case of the ALM model, a less spectacular reduction of model size would already reduce the computational effort considerably because it would imply a reduction of the number of binary variables.

The Design of an Importance Sampling Distribution

The most critical aspect of applying importance sampling is the design of the importance sampling distribution q . It has been demonstrated in 6.2.1 that the optimal importance sampling density has the property that $q_s = p_s c(\xi_s) / E_p[c(\xi)]$. However, $c(\xi_s)$ and $E_p[c(\xi)]$ are unknown. Therefore, we resort to estimating $c(\xi_s)$ and $E_p[c(\xi)]$, and q will be defined by:

$$q_s = p_s \frac{c^*(\xi_s)}{E_p^*[c(\xi)]} \quad (171)$$

The problem of specifying q has now been converted in a problem of determining $c^*(\xi_s)$ and $E_p^*[c(\xi)]$. If it is possible to specify the desired estimators analytically, then one can sample directly from the importance sampling distribution, for instance by acceptance/rejection sampling.

When one is unable to specify the importance sampling distribution analytically, one can generate a large sample, indexed $s=1, \dots, S^*$, with density function $p_s = 1/S^*$ that represents the original sample space $\{\xi_1, \dots, \xi_{S^*}\}$. Given this sample, the knowledge of the problem at hand can be exploited in order to arrive, in any computationally affordable way, at estimates $c^*(\xi_s)$ for $s = 1, \dots, S^*$. Then, the optimal importance sampling distribution can be estimated by

$$\begin{aligned}
 q_s &= \frac{1}{S^*} \frac{c^*(\xi_s)}{\frac{1}{S^*} \sum_{s'=1}^{S^*} c^*(\xi_{s'})} \\
 &= \frac{c^*(\xi_s)}{\sum_{s'=1}^{S^*} c^*(\xi_{s'})}
 \end{aligned} \tag{172}$$

Notice that, given the way in which $E_p^*[c(\xi)]$ is computed, it suffices to estimate the relative contribution of $c(\xi_s)$. If $c^*(\xi_s)$ is an optimal estimator, then $kc^*(\xi_s)$ is also optimal for any k , $k \neq 0$.

An importance sample of size S can now be obtained by sampling from density function q , which has probability space $\{\xi_1, \dots, \xi_{S^*}\}$.

6.3 Importance Sampling in ALM

In this section, we shall present an importance sampling method to reduce the size of $Twostage(t)$, the two stage model that has been formulated at page 93, subject to preserving sufficiently reliable outcomes.

Response Variables and Stochastic Input in Twostage(t)

In 6.2.1, ξ denoted the stochastic input variables and $c(\xi)$ denoted the derivative random variable of which the mean was to be estimated. In relation to $Twostage(t)$, ξ is associated with the randomly sampled states of the world (recall that they are exogenous to the model) at time t . More precisely, ξ_s contains the realisations r_{its} , L_{ts} and l_{ts} . Suppose that a sufficiently large sample of states of the world, sampled from (46), is given. Let this sample contain S^* states of the world at time t . Given the sampling procedure, the probability of state (t, s) to occur is $1/S^*$. Here, it is the intention to design conditional importance sampling distributions $q[(t, s) | (t-1, \delta)]$ which specify the importance sampling probability of (t, s) being the state that succeeds $(t-1, \delta)$.

Formulating Twostage(t) when Applying Importance Sampling

The states of the world at time t will be sampled from an importance sampling distribution $q[(t, s) | (t-1, \delta)]$. The design of $q[(t, s) | (t-1, \delta)]$ is the subject of the next paragraph. First, let us examine the procedure to formulate $Twostage(t)$, including a provision to apply importance sampling. The way in which an appropriate importance sampling distribution can be obtained will be discussed later. A stepwise description is given at page 103.

Formulating Twostage(t) when Applying Importance Sampling, a Stepwise Description

Given states of the world $(0, 1)$ and $(t-1, \hat{s})$ for $\hat{s} = 1, \dots, S_{t-1}$:

For $\hat{s} = 1, \dots, S_{t-1}$ do

- Sample $n_{t-1, \hat{s}}^*$ states of the world, indexed $(t, 1), \dots, (t, n_{t-1, \hat{s}}^*)$, from (46), each of which succeeds $(t-1, \hat{s})$ with probability $1/n_{t-1, \hat{s}}^*$.

- For $s = 1, \dots, n_{t-1, \hat{s}}^*$ estimate m_s , the relative contribution of state (t, s) to the optimal objective function. The way in which these estimates can be obtained will be discussed of 104.

- For $s = 1, \dots, n_{t-1, \hat{s}}^*$ define $q[(t, s) | (t-1, \hat{s})] = m_s / \sum_{s'=1}^{n_{t-1, \hat{s}}^*} m_{s'}$.

- Sample $n_{t-1, \hat{s}}$ states of the world from the conditional importance sampling density $q[(t, s) | (t-1, \hat{s})]$. The unconditional importance sampling probability of (t, s) to occur equals $q_{ts} = p_{t-1, \hat{s}} q[(t, s) | (t-1, \hat{s})]$

Formulate and solve the importance sampling variant on *Twostage(t)* that is presented below.

Adjusting the Formulation of Twostage(t) to Accommodate Importance Sampling

In order to obtain unbiased results with respect to the probability distribution of states of the world that has been defined by (46), the computation of probabilities and expectations in *Twostage(t)* has to be adjusted: they should be equipped with correction factors p_{ts}/q_{ts} to account for the fact the sample of states of the world at time t has been drawn from the importance sampling distribution instead of from the distribution that is defined by (46). It follows that the objective function should be changed into (173) and (142) has to be replaced by (174).

Minimise $V = A_0 + Y_{01} +$

$$\sum_{s=1}^{S_{t-1}} p_{t-1,s} \gamma_{t-1,s} (a_{t-1,s} [Y_{01} + \Delta y W] + Y_{t-1,s}) + \quad (173)$$

$$\sum_{s=1}^{S_t} \frac{p_{ts}}{q_{ts}} \gamma_{ts} (-2k_1 \theta_1 \sum_{i=1}^N (X_{its} - X_{its}^m) + \lambda Z_{ts})$$

$$\sum_{s=1}^{S_t} \frac{p[(t,s)|(t-1,\hat{s})]}{q[(t,s)|(t-1,\hat{s})]} f_{t,s} \leq \Psi_{t-1,\hat{s}}^u \quad (174)$$

The Specification of an Importance Sampling Distribution

Given a set of states of the world $(t, 1), \dots, (t, n_{t-1,s}^*)$ that succeed state $(t-1, \hat{s})$ with probability p_s , the aim is now to define a probability density function q on the probability space that consists of the aforementioned states of the world. More precisely, a *Twostage*(t)-specific version of (172) has to be specified.

In line with the argument that led to (172), the importance sampling probability of (t, s) will be based on an estimate of its contribution to the optimal objective function value of the ALM model, relative to the contribution of other states that succeed $(t-1, \hat{s})$:

$$q[(t,s)|(t-1,\hat{s})] = \frac{m_s}{\sum_{s':\hat{s}'=\hat{s}} m_{s'}} \quad (175)$$

where m_s equals the estimated relative contribution to the objective function.

To assess the contribution of state (t, s) to the expected costs of period t , relative to the contribution of other states at the world that succeed $(t-1, \hat{s})$, a mean variance efficient estimate of the optimal asset mix in $(t-1, \hat{s})$ is obtained from *MinAssets*($t-1, \hat{s}$) (specified at page 85). Given $x_{i,t-1,s}^m$, the value of the investments at the beginning of period t will be increased by a factor $\exp[r(A_{t-1,s})_{ts}] = \sum_{i=1}^N h_{its} x_{i,t-1,s}^m$ at the end of period t , if state of the world (t, s) occurs. In order to arrive at (t, s) with an asset portfolio of which the value is equal to the value of the remaining liabilities, the value at

the beginning of period t should be equal to $\exp[-r(A_{t-1,s})] \alpha L_{ts}$. Thus, it determines the minimum asset level to start with in order to meet the funding requirement at (t, s) . Based on this choice of asset mix, the weighting factor to specify the importance sampling probability of state (t, s) is computed by:

$$m_s = \exp[-r(A_{t-1,s})] \alpha L_{ts} \quad (176)$$

I.e., m_s is the minimal asset value that is required to be invested in $(t-1, \hat{s})$ in order to satisfy the solvency requirement in (t, s) given that the assets have been invested in the mean variance optimal portfolio according to $MinAssets(t-1, \hat{s})$. It is conceivable, however, that the optimal policy requires funding levels that exceed αL_{ts} . As long as these funding levels can be attained by increasing the initial asset level and maintaining the asset mix that has been selected by $MinAssets(t-1, \hat{s})$, $q[(t, s) | (t-1, \hat{s})]$ remains a good importance sampling density, because it is only affected by changes in the relative contributions to the costs, not by changes of the level of expected costs.

Given m_s , $q[(t, s) | (t-1, \hat{s})]$ is computed by normalizing $q[(t, s) | (t-1, \hat{s})]$ so that the summation of $q[(t, s) | (t-1, \hat{s})]$ over all possible states of the world adds up to 1.

Notice that m_s has been constructed so that, given positive liabilities, $m_s > 0 \forall s$ which implies that q_s , as specified above, is indeed a density function.

The computational results in 6.4 provide insight in the relationship between variance of the response variables, the sample size when sampled from the original distribution and the sample size when using importance sampling with the choice of importance sampling distribution that has been described above.

6.4 Computational Results

In order to obtain insight in the practical use of importance sampling, *TEST_IS*, a one period chance constrained programming model, has been solved by three methods:

- by quadratic programming,
- by scenario optimisation, based on scenarios that have been obtained from random sampling from the underlying distribution,
- by scenario optimisation, based on scenarios that have been obtained from importance sampling, using the importance sampling distribution that has been described earlier.

TEST_IS

$$\text{Minimise } A_0 - \gamma(E[A_T] - 1) \quad (177)$$

s.t.

$$\sum_{i=1}^N X_{i0} = A_0 \quad (178)$$

$$E[A_T] = \sum_{i=1}^N (1 + E[r_i])X_{i0} \quad (179)$$

$$\sigma^2[A_T] = \sum_{i,j=1}^N X_{i0} \sigma[r_i, r_j] X_{j0} \quad (180)$$

$$E[A_T] \geq 1 \quad (181)$$

$$(E[A_T] - 1)^2 \geq (\varphi^{-1}(\Psi^u))^2 \sigma^2[A_T] \quad (182)$$

$$X_{i0} \geq 0 \quad (183)$$

This problem can be interpreted as a simplified ALM problem: the expected present value of the costs of funding a liability to the amount of 1 has to be minimised by choosing an asset mix and an initial level of investments. The costs of funding should be minimised subject to the probability of not being able to meet the liability being less than or equal to Ψ^u , under the assumption that asset returns follow a joint normal distribution with a covariance matrix and vector of expected returns in conformity with Table 6. The upper bound on the probability of underfunding has been set at 0.05.

The scenario optimisations have been conducted for sample sizes equal to 25, 50, ..., 250. For each sample size, the problem has been solved for 50 samples. Because of the assumption that the returns follow a joint normal distribution, an optimal solution to the problem could be obtained by quadratic programming. Given this optimal solution and the solutions to each of the stochastic programming problems, standard errors have been computed for the initial asset mix, the expected present value of costs, the level of the initial investments. Moreover, given the initial level of investments and the initial asset mix that follow from the stochastic programming solutions, it has been determined what the true probability of underfunding, based on the underlying continuous probability distribution of returns, is that would be associated with the each of stochastic programming solutions.

Figure 12 - Figure 15 present the standard errors, expressed as a percentage of the optimal value, as a function of the sample sizes. As can be verified from the graphs, the overall accuracy of the solutions that are based on importance sampling, measured by the standard errors of the values of the relevant decisions variables, is better than those obtained from random sampling. In particular, the probability of underfunding is estimated much better from the importance samples than from the random samples. The interested reader can be referred to Aarssen (1992) for more extensive empirical results on importance sampling in ALM.

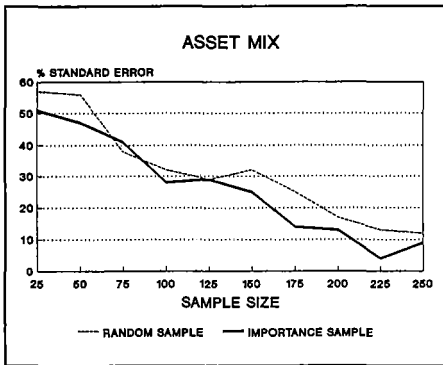


Figure 12

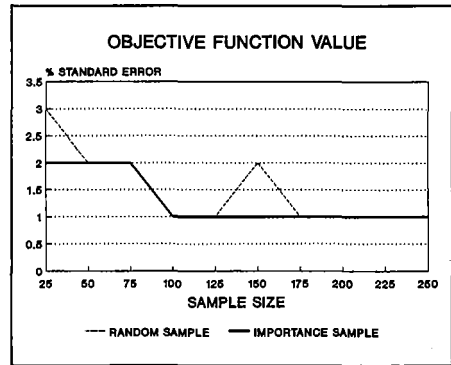


Figure 13

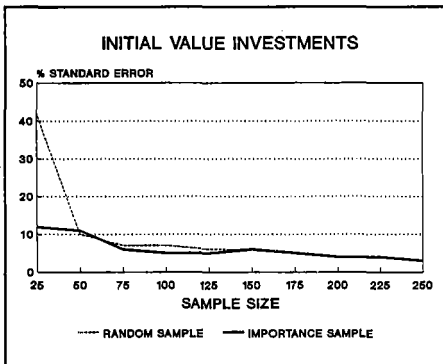


Figure 14

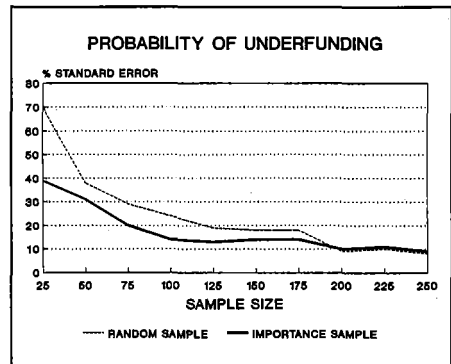


Figure 15



Chapter 7

Computational Results

7.1 Introduction

This chapter presents results of computational experiments with the ALM model. In order to obtain insight in the behaviour of the model on realistic problem instances, it has been applied to the data of a Dutch pension fund with an actuarial reserve in excess of 16 billion Dfl. and approximately 1,020,000 participants of which 240,000 premium payers.

Computational results on a matter as complex as dynamic ALM may give rise to many questions from different viewpoints. In particular, it offers a host of possibilities for sensitivity analyses: with respect to decision parameters, with respect to economic scenarios, with respect to the historical period and with respect to the estimation technique that has been used to obtain coefficients for the scenario generator, with respect to the characteristics of the participants of the pension fund, with respect to the financial starting position of the pension fund, with respect to the asset categories that have been selected to invest in, with respect to the optimisation parameters, with respect to the horizon, with respect to further future lagging of the state pensions, with respect to the employment of derivatives etc. It is beyond the scope of this monograph to pay attention to all the questions that could be raised from the above viewpoints, however interesting many of them may be. At this stage, the focus will be on experiments that give insight in the behaviour of the ALM model, in comparison to some of the leading models that are currently being used. Moreover, results will be presented to show to what extent solutions are sensitive to small changes in the most relevant exogenous parameter values.

The selection of computational experiments has been made in such a way that the results provide insight in the following questions:

1. How do the results of our ALM approach compare to the results that follow from making optimal static decisions ?
2. To what extent are the solutions to the ALM model driven by its multistage character ? In other words, how does the backward procedure affect the outcomes ?
3. How sensitive is the solution to small changes in the most relevant exogenous parameter values ?

The next section discusses the environment within which ALM policies will be determi-

ned. The results of the computational experiments will be presented in 7.3.

7.2 The Exogenous Environment

The exogenous environment within which the ALM policies have to be determined is specified by the coefficients of the scenario generator, the initial state of the world and the requirements on the ALM policies. These will be discussed in the following paragraphs.

7.2.1 The Scenario Generator

Before turning to the coefficients which have been used for the scenario generator, we shall give a statistical overview of the historic time series that are of importance.

Historical Time Series

Consider the statistics in Table 6. They have been computed on the basis of annual observations of the following time-series: Dutch inflation of wages, Dutch inflation of prices, the return on short term deposits denominated in Dutch guilders, total returns on

Table 6. Statistics based on annual observations from primo 1956 until ultimo 1994.

	wages	prices	cash	stocks	gnp	property	bonds
Average	5.68%	4.40%	6.02%	10.24%	3.27%	7.67%	6.38%
standard deviations and correlations							
wages	3.98%						
prices	0.52	2.81%					
cash	-0.17	0.28	2.45% ²⁶				
stocks	-0.21	-0.24	-0.16	16.77%			
gnp	0.33	0.14	-0.31	-0.34	2.34%		
property	0.17	0.16	-0.06	0.33	-0.25	8.12%	
bonds	-0.07	-0.05	0.19	0.43	-0.49	0.52	8.16%

an internationally diversified stock portfolio (Robeco), growth of Dutch gross national product, total returns on an internationally diversified property portfolio (Rodamco) and total returns

²⁶ Although the time series of returns on short term deposits has a positive standard deviation, the return on annual deposits over the next year is known with certainty before the asset allocation decision has been made.

on an internationally diversified bond portfolio (Rorento). Sorting the investment categories from high to low, according to average return, gives the following order: 1. stocks, 2. property, 3. bonds, 4. cash. Notice, however, that the average return on bonds only just exceeds that on cash. Sorting the asset classes according to standard deviation of return yields the same sequence, with the exception of property and bonds. It is not out of the question that the relatively low standard deviation of property is partly due to valuation issues. It is interesting to see that the return on cash is only slightly lower than that on bonds, whereas the standard deviation of the return on bonds is more than three times as high as the standard deviation of the return on cash. Moreover, when the growth of liabilities is (partially) determined by price inflation, then the high correlation of the return on cash with price inflation makes cash, as compared with bonds, a very attractive asset class.

Table 7. Autocorrelations based on annual observations from primo 1956 until ultimo 1994.

	wages	prices	cash	stocks	gnp	property	bonds
wages	0.33	0.52	-0.21	-0.34	0.47	-0.04	-0.28
prices	0.36	0.73	0.17	-0.19	0.22	0.07	-0.23
cash	-0.20	0.24	0.66	-0.06	-0.17	-0.03	-0.04
stocks	0.09	-0.03	0.33	-0.09	-0.32	-0.20	0.11
gnp	0.13	-0.16	-0.63	0.26	0.40	0.05	-0.22
property	0.12	0.30	0.12	-0.25	-0.03	-0.35	0.04
bonds	-0.14	0.21	0.59	-0.08	-0.27	-0.36	0.17

Table 7 presents autocorrelations on annual data. Using the econometric rule of thumb that autocorrelation coefficients are statistically significant if their absolute value exceeds $2/\sqrt{n}$, where n denotes the number of observations, one should be particularly interested in figures that exceed 0.33 in absolute value. Most figures may not come as a surprise. However, there is an exception: the correlation between the return on stocks and the lagged return on cash is rather high. If this statistical relationship tells us anything about the future, then the return on Dutch deposits can be used to predict the return on internationally diversified stock portfolios. Despite this interesting observation, we have chosen not to include this relationship in the set of a priori expected relationships.

Historical Time Series and ALM

To appreciate the trade-off that has to be made in ALM, let us go one step further. Suppose that liabilities are fully indexed with price inflation and that annual contributions to the fund

are precisely sufficient to fund newly acquired pension rights, exclusive of the increase of liabilities that is due to indexation promises. Then, in order to maintain the existing funding level, the average return on investments may not be less than $4\% + \text{average price inflation} = 8.4\%$. To compose an asset mix with such a high average return, one has to invest substantially in equities: property and stocks are the only asset classes with an average return that exceeds 8%. Such an asset mix, however, is also characterised by a high standard deviation of return, which implies that it entails an unacceptably high probability of underfunding in the shorter term, unless the sponsor is willing and able to accept large fluctuations of annual contributions, or the pension fund has a high funding level to start with. Otherwise, it cannot afford the risk of sustaining substantial losses on the investment portfolio; the fund would quickly become underfunded.

Estimated Coefficients of the VAR Model

The numerical results in this paper have been obtained by estimating the parameters of (46) by applying the iterative SUR method to annual observations over the period 1956 - 1994. Table 8 presents the relationships that were expected a priori. A + indicates that a coefficient with positive sign was expected, a - indicates that a negative coefficient was expected. As has been explained in 4.3.1, these a priori expectations have been used to determine the order by which statistically insignificant coefficients are omitted from the system of equations. The values of the estimated coefficients are given in Table 9. Standard errors of the estimated coefficients are given in brackets. Recall that the model to which these figures belong has been specified in terms of continuous returns, inflations and growth of gross national product.

Table 8. Expected relationships

dependent variable	independent lagged variable			
	prices	cash	gnp	bonds
wages	+		+	
prices	+		+	
cash	+	+	+	
stocks				
bonds	-		-	
property	+			
gnp		-		+

Given a state of the world, the succeeding states of the world are defined by a deterministic component that reflects the expected development, conditional on the current state of the world, and a random component that is sampled from a distribution with expectation 0 and a covariance matrix that is determined by the correlations and standard deviations that are presented in Table 10.

Table 9. Estimated coefficients and standard errors of the VAR model

dependent variable	independent variables			mean	adjusted R-squared
	intercept	price inflation	cash		
wage inflation	0.026929 (0.00828)	0.654292 (0.165032)		0.055069	0.24
price inflation	0.014001 (0.005152)	0.653854 (0.095871)		0.042115	0.49
cash	0.019525 (0.006053)		0.679611 (0.092472)	0.059059	0.42
stocks	0.084692 (0.024570)			0.085635	0.00
property	0.071748 (0.017342)			0.072410	0.00
bonds	-0.035571 (0.022893)		1.634033 (0.349911)	0.060482	0.27
GNP	0.062338 (0.006944)		-0.525310 (0.107974)	0.031594	0.36

Table 10. Residual correlations and standard errors of regression

	wages	prices	cash	stocks	gnp	property	bonds
wages	0.03						
prices	0.28	0.02					
cash	-0.12	0.32	0.02				
stocks	-0.23	-0.31	-0.53	0.16			
gnp	0.34	0.40	0.20	-0.18	0.02		
property	0.04	-0.02	-0.19	0.33	-0.22	0.11	
bonds	-0.01	-0.27	-0.33	0.35	-0.20	0.55	0.07

The Initial State of the World

Table 11 contains data on the pension fund as at the beginning of 1995, after benefits and contributions for 1995 have been paid. Notice that the actuarial reserve amounts to 4 times the pensionable earnings and more than 20 times the annual contribution to the fund. This implies that the contribution policy will have only a limited impact on the

development of the fund in comparison with the investment results.

Table 11. The pension fund as at the beginning of 1995.

	active participants	inactive participants	total
number	240,000	780,000	1,020,000
benefit payments '95		Dfl. 300 mln.	Dfl. 300 mln.
contribution '95	Dfl. 700 mln.		Dfl. 700 mln.
pensionable earnings		Dfl. 4,100 mln.	Dfl. 4,100 mln.
indexation promise	wage inflation	price inflation	
actuarial reserve	Dfl. 7,600 mln.	Dfl. 8,800 mln.	Dfl. 16,400 mln.

The initial state of the world is also of importance for the scenario generator: it contains the starting values of the economic variables. These values are given in Table 12.

Table 12. Values of the economic variables over 1994.

wage inflation	price inflation	return on				
		cash	stocks	gnp	property	bonds
1.6%	2.6%	5.12%	-7.10%	2.4%	-20.75%	-15.52%

The Future According to the Scenario Generator

To facilitate the interpretation of the ALM strategies, consider Figure 16, Figure 17 and Table 13. They present the average development of cumulative investment returns of the asset categories and cumulative growth of pensionable earnings, benefit payments and actuarial reserves. A comparison of the average investment returns with the average growth figures of liabilities and benefit payments shows that, on average, liabilities will be increased more rapidly than the value of invested assets.

In order to prevent annual probabilities of underfunding to exceed 5%, ALM strategies will have to rely on contributions that exceed benefit payments substantially, or on asset mixes that yield an average return which does not lag too much behind to the growth of liabilities.

An ALM strategy can cope with this phenomenon by securing a level of regular contributions that exceeds benefit payments to an extent that suffices to compensate for the slow appreciation of the value of invested assets, relative to the growth of liabilities. This will only be a realistic policy, however, as long as the ratio of the value of invested assets and the level of pensionable earnings is reasonably small. In other words, as long as the fund is relatively immature. Otherwise, the level of contributions, as a percentage of the pensionable earnings that is required to make up for the lagging investment returns rises dramatically. Of course, the smaller the difference between investment returns and growth of liabilities, the less serious this problem is. Therefore, a pension fund should prefer to select an asset mix with a high expected return. However, to bear the risk of substantial negative investment returns that comes with such an asset mix, the fund must have an initial surplus that can serve as a buffer for the volatility of the investment returns. This calls for a high funding level to start with.

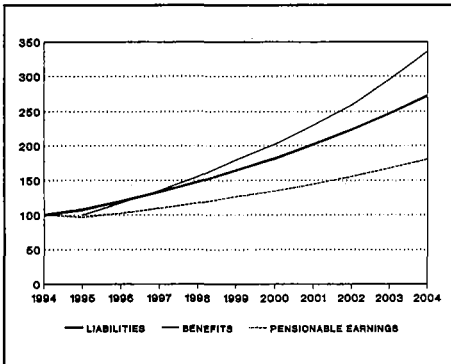


Figure 16. Cumulative expected growth

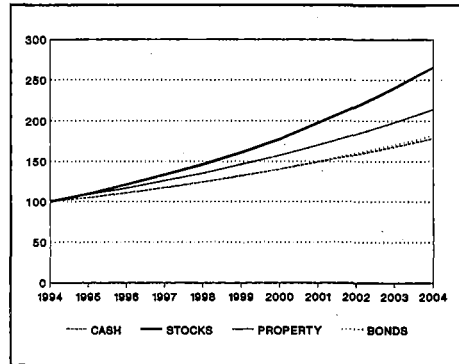


Figure 17. Cumulative expected return

Table 13. Sample statistics 1995-2004

	annual growth			annual investment returns			
	liabilities	benefits	pens. earnings	cash	stocks	property	bonds
Average	10.6%	5.1%	3.8%	6.0%	10.3%	8.0%	6.2%
Standard deviation				2.0%	15.8%	10.6%	7.0%

In summary, the scenarios reflect an expected growth of liabilities that is rather high in comparison with the expected return on the asset classes. Therefore, feasible ALM strategies will have to maintain high funding levels in order to be able to bear the risk that comes with the selection of asset mixes with a high expected return. If the appropri-

ate funding levels cannot be attained, then solvency requirements can be met only by dramatic raises of annual contribution. The latter becomes worse with the extent to which the fund has matured.

7.2.2 Requirements on the ALM Policy

ALM policies will be determined under the constraints and parameter values that are given in Table 14. The horizon has been chosen equal to 10 years. Although the choice of horizon may affect the optimal decisions, computational results on problems with longer horizons do not provide more insight in the model. From a computational point of view, recall that the running time to solve the ALM model depends linearly on the number of stages. Therefore, if one is interested in analyzing a pensionfund in an environment of which the characteristics are expected to be changing significantly after 10 years, then the horizon can be extended as appropriate, without incurring computational problems.

Table 14. Model parameters and constraints

contribution level for 1995	16% of the pensionable earnings
minimally required funding level	100%
maximum probability of underfunding	0.05
upper bound on proportional asset allocation	1
lower bound on proportional asset allocation	0 (no short selling)
horizon	10 years
number of states of the world at the end of the first year	100
number of states of the world at the end of year 2, ..., 10	10,000
discount rate	15% per annum

Different Settings, Different Policies

One would expect ALM decisions for a wealthy pension fund to be different from those for a thinly funded pension fund. Therefore, three settings have been selected, which differ in the initial funding level and in the amount by which annual contributions may be raised from one year to the next:

- Setting 1: a low initial funding level and a low maximum increase of contributions,
- Setting 2: a high initial funding level and a high maximum increase of contributions,
- Setting 3: the initial funding level to be determined by the ALM model in such a way that the costs of funding are minimised subject to satisfying the solvency constraints without exceeding moderate maximum increases of contributions.

In addition to the data that have been specified in Table 14, Table 15 presents setting-specific constraints:

Table 15. Setting - Specific Data

	Setting 1	Setting 2	Setting 3
Maximum raise of annual contributions if new level > 16%	2 percent points	5 percent points	3 percent points
if new level ≤ 16%	unrestricted	unrestricted	unrestricted
Asset value primo 1995	17,900 mln	32,800 mln	To be determined by the ALM model
Funding level primo 1995	109%	200%	

The settings have been selected with the intention to analyze the ALM model on problems of which it may be expected that the driving force behind their solutions will be different for each problem.

A quick analysis of the expected development of the actuarial reserve, benefit payments and the pensionable earnings, in combination with the maximum growth of contributions reveals that a feasible ALM strategy for setting 1 may not exist. In this case, the solution will be dominated by the search for a solution that minimises the probabilities of underfunding, while building up a sufficiently high funding level.

In setting 2, on the contrary, a high initial funding level has been specified, in combination with a relatively high maximum on annual increases of contribution. In this setting, one would not expect any problems with respect to meeting the solvency requirements. Here, the issue is the trade-off between reducing short term costs by making large restitutions in the first years and maintaining a high funding level in order to preserve the

ability to meet solvency constraints, and to reduce longer term costs.

If the fund has insufficient financial elbow room in setting 1, and excessive wealth in setting 2, then the question arises what the optimal initial funding level and the corresponding ALM strategy should be. In other words, how much should be contributed to the fund, or may be withdrawn from the fund, such that it can meet the solvency requirements with moderate hikes of contribution only, at minimal expected costs of funding. That problem is reflected by setting 3.

In practice, one may want to impose other constraints, such as upper and lower bounds on the proportional allocation of assets, as well. Although the ALM model has been developed to accommodate other constraints as well, we have chosen to present computational results which are driven by policy constraints as little as possible in order not to veil the main forces that drive the ALM model.

7.2.3 Comparison of the ALM Model to Other Approaches

One of the most interesting questions that remains is the extent to which our ALM approach yields decisions that differ from other types of models and how the results compare.

Dynamic Stochastic Programming Approaches

It is difficult to compare our results to other scenario approaches: to our knowledge there have been no publications of computational results on a widely accepted set of test problems. Cariño et al (1993,1994) report on computational results of a 5 stage stochastic programming model that uses a small number of states of the world to describe the development of an environment that contains more than 5 stochastic state variables. This has been discussed in chapter 2. Dempster and Corvera Poiré (1995) have developed a stochastic programming technique on which they have presented computational results on 10 stage problems with up to 2000 states of the worlds. In this chapter, we present results on 10 stage problems, taking into account more than 100,000 states of the world. One should be cautious, however, not to attach too much value to the number of states of the world that is taken into account: the employment of different variance reduction techniques may imply that one approach needs a greater number of states of the world than other approaches, in order to obtain a similar representation of the uncertain development of the exogenous environment. As a consequence, it is difficult to assess how well the stochastic environment is modelled when only the number of states of the world is given.

A more fundamental difference between the aforementioned models and our model, is the fact that we employ endogenous binary variables to register in which states of the world underfunding occurs. This enables us, in contrast with the other approaches, to include probabilities of underfunding in each state explicitly. Moreover, the fact that these probabilities have been defined as a function of events that happen in the scenarios, implies that it is not necessary to specify the probability distribution of the states of the world at various times explicitly. The optimisation model accommodates any probability distribution that can be reflected by the set of states of the world.

Static Approaches

To compare our results to the results from static models, a static asset mix has been determined, in combination with a maximum funding level and a minimally desired funding level. These funding levels serve to determine state dependent contribution levels by the decision rules that are specified in Table 16.

Table 16. Static Contribution Policy

actual funding level	contribution level
1. $>$ maximum funding level	restitution such that the maximum funding level remains
2. $>$ minimum funding level and $<$ maximum funding level	16% of the pensionable earnings
3. $\geq 100\%$ and $<$ minimum funding level	minimum of the contribution level that is required to restore the minimum funding level and the maximum contribution that follows from the constraint on annual rises of contribution
4. $< 100\%$	remedial contribution to restore 100% funding level and regular contribution as in 3.

The optimal asset mix and the optimal values of the funding levels below which regular contributions are increased and above which restitutions are made, have been determined by a brute force random search method. To our knowledge, the static models in the literature, this includes mean variance approaches, do not allow for more flexible decision rules than the ones that have been used here. Therefore, the solutions that have been obtained from this static approach, will be at least as good as solutions that can be obtained from static models from the literature.

The Contribution of Dynamic Decisions to the Results from the ALM Model

It has not been possible to prove **the effectiveness of the backward procedure** analytically. Recall that it serves to collect and transfer information on years to come in order to guide decision making at earlier instants. To obtain insight in the effect that the backward procedure has on the solutions to the ALM model, the heuristic by which we solve the ALM model has also been run without the backward procedure. In that case, the forward procedure remains. This method will be referred to as **forward-only**. Without the input from the backward procedure, the forward procedure minimises expected costs of funding in each state of the world, subject to satisfying solvency constraints that follow directly from the level of actuarial reserves at the succeeding states of the world. This method simulates a policy where optimal time and state dependent decisions are made, taking into account a one year horizon only. The differences between the results from the forward-only procedure and the proposed ALM approach can be attributed to the backward procedure.

7.3 Numerical Results

7.3.1 The Behaviour of the ALM Model

Which Policy is the Better One ?

The ALM policies will be judged on three criteria: stability of regular contributions, the extent to which solvency requirements are met and the present value of the costs of funding.

The stability requirements with respect to regular contributions are always met, by each policy, in each setting: the level of regular contributions is set at the beginning of each year, such that it satisfies the upper bound that follows from the maximum raises of contribution and the regular contribution level that has been set in the preceding year. If it turns out, by the end of the year, that previous decisions have led to a situation of underfunding, then a remedial payment is made to restore the minimally required funding level.

The solvency requirements are reflected by upper bounds on the annual probability that underfunding occurs. In each scenario, at the end of each year the funding level is computed, before remedial contributions are made. If it is less than 100%, it counts as a situation of underfunding. A policy is feasible if the solvency requirements are met. In other words, it is feasible if the probability of underfunding is less than or equal to 5% in

each year. The extent to which this requirement is met is reflected by the **average excess probability of underfunding**. Feasible policies have an average excess probability of underfunding that is equal to 0. The extent to which a policy leads to situations of underfunding can also be measured by the magnitude of deficits when they occur. This is reflected by the present value of expected remedial payments.

The requirements on the ALM policies and the objective to minimise the present value of expected costs of funding imply that, given two sets of results, the following stepwise analysis should be used to determine which of the associated policies is the better one:

Step 1. Feasibility

- If both policies are feasible go to Step 2.
- If both policies are infeasible, then the one with the lower average excess probability of underfunding is the better policy.
- If one of the policies is feasible and the other is not, then the feasible policy is the better one.

Step 2. Costs of funding

- The policy with the lower **present value of total costs** is the better one.

7.3.2 Results From the Forward-only Procedure

The main results from the forward-only procedure are presented at page 126. In setting 1, the low initial funding level leads to the selection of an asset mix with a low volatility of investment returns. Riskier asset mixes cannot be chosen because it would imply too high a probability of underfunding by the end of 1995. The latter argument holds for years to come as well: regular annual contributions may be raised by only 2 percent points of the pensionable earnings which is insufficient to increase the funding level substantially. The average return on the 'safe' asset mixes, however, is lower than the growth of liabilities. These effects result in unacceptably high probabilities of underfunding.

The initial asset mixes in settings 2 and 3, 100% stocks, reflect the attractiveness of a high expected investment return, combined with a surplus that is sufficiently high to bear the risk of unfavourable investment returns. More interesting, however, is the greedy character of the forward-only procedure that becomes manifest in the results in Settings 2 and 3. Instead of making decisions that preserve the high initial funding levels to ensure solvency in the longer term, large restitutions are made during the first years of the planning period. As a consequence, it takes only a few years, after which the fund arrives in a situation that is comparable to the starting situation in setting 1: a low funding level

from which it is difficult to recover.

The results of the forward procedure are a clear illustration of the risk of following a policy that is driven by short term results only: the price for short term reductions of the costs of funding is paid by arriving in an arduous situation from which it is difficult to recover.

In summary, due to the fact that decisions are made on a short term basis only, the forward procedure leads to infeasible policies, independent of the starting situation.

7.3.3 Results From Optimal Static Decisions

Page 127 displays the main results from the static model. The static model differs from the forward-only procedure in two important respects. The forward-only procedure determines state dependent and time decisions whereas the static model makes all decisions at the beginning of the planning period, independently of the scenario that will materialize. Thus, in case of the static model, the initial asset mix is the asset mix that will be held in all future states of the world, irrespective of the prevailing funding level. The other main difference is the fact that the static decisions do take into account the entire planning period, whereas the results of the forward-only model were substantially affected by the fact that it only reckons with a horizon of one year.

As one would expect, the results of the static model are to a large extent determined by the inability to make state dependent decisions. This is reflected by the choice of asset mixes, as well as by the development of the probability of underfunding. This effect is illustrated particularly well by the results in setting 1. The optimal asset mix is too risky to meet short term solvency requirements: ultimo 1995, the probability of underfunding equals 15% and it rises to a dramatic 28% in 1999. Thereafter, it stays high but it starts to decline. This due to the fact that, by then, the increased level of contributions in combination with the fact that the asset mix which does provide an investment return that, in combination with the increased contribution level, suffices to ensure a gradual rise of the average funding level. The high probabilities of underfunding that occur despite the fact that the average funding level exceeds 120% towards the end of the planning period, reflect the limitations of the static model: the asset mix is too risky in states of the world with a low funding level and too conservative in states of the world with high funding levels. The forward-only model performed much better in this setting.

In settings 2 and 3, the optimal asset mixes are on the conservative side given the high funding levels during the first years. At the end of the planning period, however, the

probabilities of underfunding rise steadily which result in slight violations of the solvency constraint for 2003 and 2004 in setting 2; the asset mix which may have been too conservative in earlier years, proves to be too risky to be selected in all states of the world towards the end of the planning period.

The static policies remind one of a man who stands in the blazing sun in ice cold water up to his hips: the average temperature is fine, but the situation is far from ideal.

In summary, the inability to react to situations that have emerged at the time of decision making, which is inherent to static models, leads to poor solutions. The optimal static decisions result in dramatically high probabilities of underfunding in setting 1. In setting 2 and setting 3, this shortcoming is demonstrated as clearly as in setting 1: the high initial funding levels and the fact that the static decisions do take the entire planning period into account, result in an almost feasible strategy for setting 2 and a feasible strategy for setting 3.

7.3.4 Results from the ALM Model

In relation to the forward-only model and the static model, the ALM model should offer best of worlds: decisions make use of the situation that has emerged at the time of decision making and the entire planning period is taken into account. The numerical results are presented at page 128.

First, let us discuss the choice of initial asset mixes. The choice for setting 1 has been driven by the solvency requirement at the end of the first year: given the low initial funding level, the asset mixes with higher expected returns are too risky. Particularly interesting is the choice of asset mix for setting 2. One would expect an initial funding level of 200% to be sufficiently high to bear the risk of the asset mix with the highest expected return, 100% stocks. And in fact, this observation is correct. However, this decision is not so much driven by the short term probability of underfunding. Instead, it is driven by the target of maintaining a funding level that, according to the backward procedure, is optimal in view of the objective of minimising the present value of expected costs of funding. This also explains why the initial funding level that has been selected in setting 3 amounts to a figure as high as 251%: it is the minimum funding level for which all assets can be invested in stocks while satisfying chance constraints on arriving at optimal funding levels at the end of 1995.

At first sight, it may seem strange that it is optimal to maintain an unnecessarily high funding level when the expected return on investments is at most 10.3% whereas the

discount rate for future contributions and restitutions is 15% per annum. This is caused by the fact that a relatively small additional investment enables one to select an asset mix which enhances the average return on the entire investment portfolio. Thus, the incremental revenues from an additional unit of investments is high in comparison with the cost of capital of this unit, the latter being reflected by the discount factor. As soon as the expected return on investments does not increase any more when the initial asset level is increased (i.e., when 100% is invested in stocks), it is not attractive any more to increase the funding level: then, the incremental revenues are only 10.3% of the additional investment whereas future restitutions will be discounted with 15%.

This mechanism can be illustrated by a simple example. Suppose that one has to choose between two asset mixes: 100% cash and 100% stocks. The return on the former being characterised by a zero standard deviation and an expected return equal to 5%, the latter by an expected return equal to 10% and a standard deviation equal to 16%. What would the minimum initial investment be that suffices to meet a fixed liability to the amount of Dfl. 100 after one year, with probability 0.95? If one assumes normally distributed returns²⁷, it can be shown that the chance constraint implies that the worst investment return for which the liability should still be met equals -16.4%. As a consequence, the minimum initial asset level that is required when the risky mix is chosen is equal to $100/0.836 = 120$. Given an initial asset level to the amount of 120, the expected level of assets at the end of the year is $1.1 \cdot 120 = 132$. Accounting for the restitution of the excess value to the amount of 32, the present value of the costs of funding is $120 - 32/1.15 = 92$. Let us compare these costs with the costs of funding that would have been incurred if one had chosen to invest 100% in cash. In that case the return on investments was known to be 5% with certainty. It follows that the minimal initial asset level is $100/1.05 = 95$, which is also the costs of funding. Thus, the present value of the expected costs of funding when the risky asset mix was chosen are lower than the expected costs of funding that are associated with the choice for the safe asset mix, despite the fact that the discount rate exceeds the return on investments by 5 percent points.

The mechanism that has been illustrated by the above example plays an important role in the ALM policies. It explains why the present value of the costs of funding in setting 3, in which a very high initial asset level has been selected is lower than the present value of the costs of funding in setting 2. Moreover, the trade-off between short term costs and longer term costs that has been made by the backward procedure generally results in

²⁷ The assumptions of normally distributed returns and a fixed liability have been made in order to simplify the presentation. This argument can be presented in a completely analogous fashion under the assumption of lognormally distributed returns.

desired funding levels that exceed the ones that are minimally required in order to satisfy solvency constraints. This can lead to optimal funding levels that increase with the length of the planning horizon.

In this respect, it is interesting to compare the development of the average funding level in setting 3 with that in setting 2. The higher initial funding level in setting 3 has been reduced rapidly by making large restitutions from 1996 until 2001. During the first years of the planning period, the ALM policy in setting 2 is geared more towards maintaining the given funding level, which results in substantially higher contributions until 1998. As of 1998, the average funding levels, as well as average contributions and probabilities of underfunding in setting 2 closely follow the development of the corresponding statistics of setting 3. The slightly lower average funding level that is maintained in setting 2, may reflect the greater flexibility in setting 2 with respect to raises in regular contributions (5 percent of the pensionable earnings in setting 2, as compared to 3 percent in setting 3).

In summary, the results from the ALM model do indeed reflect a trade-off between long term effects and short term effects.

RESULTS FROM THE FORWARD-ONLY PROCEDURE

Initial asset mix

setting	Cash	Stocks	Property	Bonds
1	58	17	25	0
2	0	100	0	0
3	0	100	0	0

Feasibility of solutions

average excess probability of underfunding (violation of chance constraints)		
setting 1	setting 2	setting 3
10%	5%	4%

Average total contr. (% of pens. earnings)

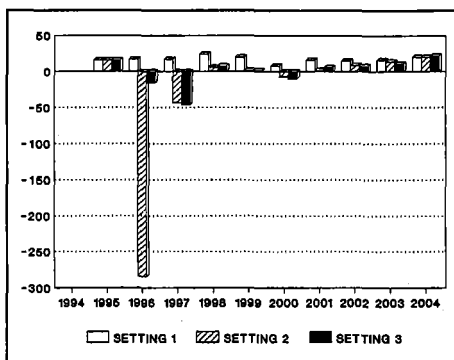


Figure 18

Average funding level

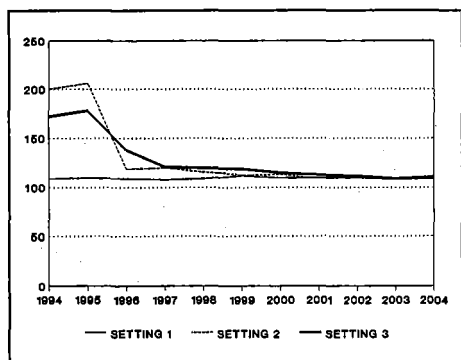


Figure 19

Probability of underfunding

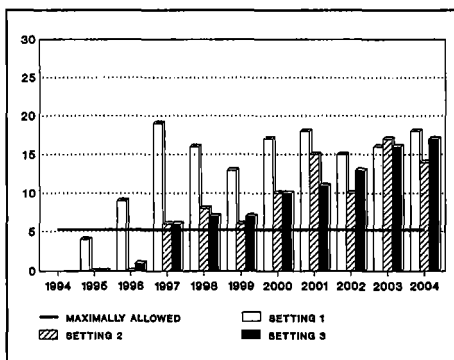


Figure 20

Table 18. Present value (PV) of the costs of funding in mln. Dfl.

setting	Initial asset level	PV regular contribution	PV remedial contribution	PV total contribution	PV terminal surplus	PV total costs
1	18,075	4652	747	5399	1019	22,455
2	33,705	-10,337	385	-9952	1312	22,441
3	29,239	-5808	415	-5393	1063	22,783

RESULTS FROM OPTIMAL STATIC DECISIONS

Initial asset mix

setting	Cash	Stocks	Property	Bonds
1	49	18	33	0
2	35	28	31	6
3	14	52	32	2

Feasibility of solutions

average excess probability of underfunding (violation of chance constraints)		
setting 1	setting 2	setting 3
17%	0%	0%

Average total contr. (% of pens. earnings)

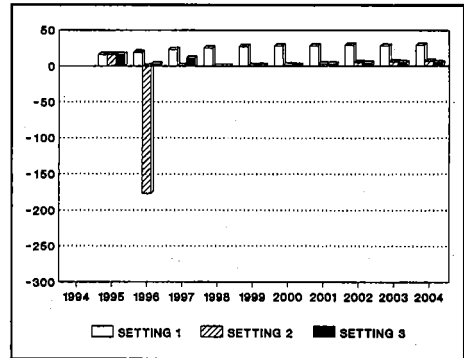


Figure 21

Average funding level

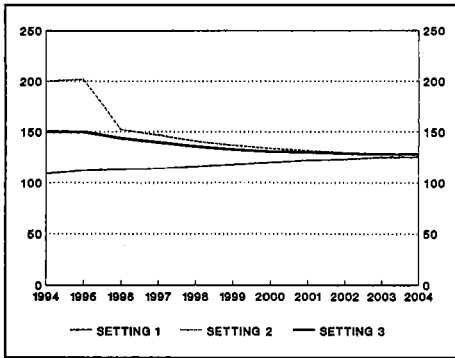


Figure 22

Probability of underfunding

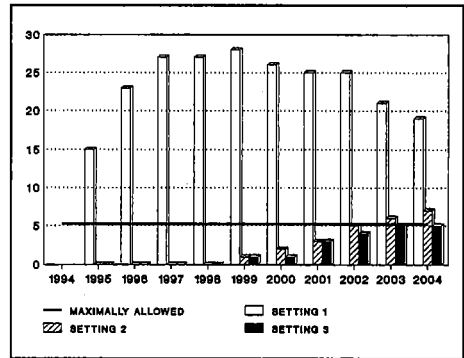


Figure 23

Table 20. Present value (PV) of the costs of funding in mln. Dfl.

setting	Initial asset level	PV regular contributions	PV remedial contributions	PV total contributions	PV terminal surplus	PV total costs
1	18,075	-16,798	24,340	7542	2711	22,906
2	33,705	-1888	1276	-612	3030	30,063
3	24,874	4484	827	5311	3086	27,099

RESULTS FROM THE ALM MODEL

Initial asset mix

setting	Cash	Stocks	Property	Bonds
1	66	21	13	0
2	52	44	4	0
3	0	100	0	0

Average total contr. (% of pens. earnings)

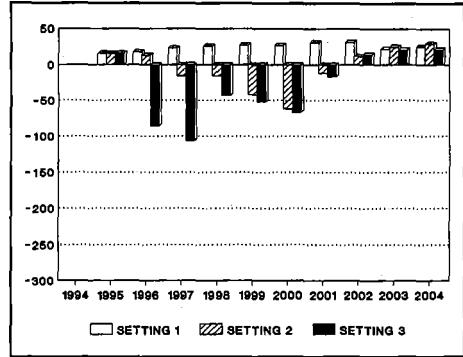


Figure 24

Feasibility of solutions

average excess probability of underfunding (violation of chance constraints)		
setting 1	setting 2	setting 3
7%	0%	0%

Average funding level

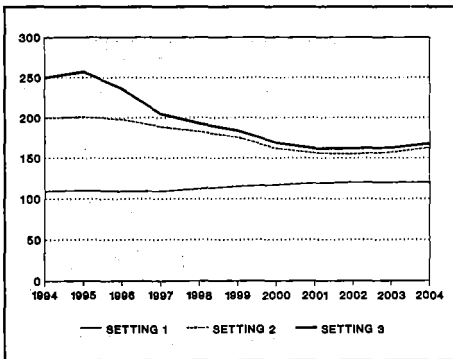


Figure 25

Probability of underfunding

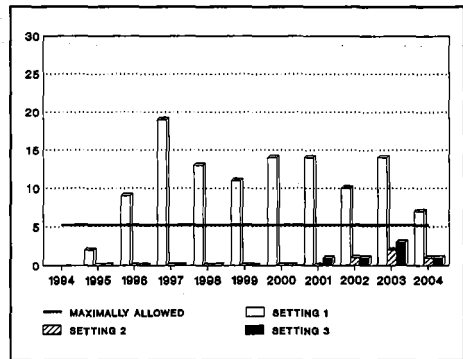


Figure 26

Table 22. Present value (PV) of the costs of funding in mln. Dfl.

setting	Initial asset level	PV regular contributions	PV remedial contributions	PV total contributions	PV terminal surplus	PV total costs
1	18,075	6494	699	7194	2266	23,003
2	33,705	-1580	19	-1561	6788	25,356
3	42,083	-10,069	42	-10,027	7374	24,682

The Extent to Which the Solutions are Affected by Small Changes in the Data

In order to assess to what extent the optimal decisions at time 0 change as a consequence of small perturbations of exogenous data, solutions to the ALM model have been computed in two additional settings. Both settings are variants on setting 3, the least restrictive one. The additional settings are defined as follows:

Setting 3a: as setting 3, but all positive elements of the autocorrelation matrix have been reduced by 100% of their standard error. Likewise, all negative elements have been increased by 100% of their standard error. The intercepts have been changed in such a way that the long term expectations remain unchanged.

Setting 3b: as setting 3, with a maximally allowed probability of underfunding equal to 6% instead of 5%.

Table 23. The effect of small perturbations of the exogenous data on the initial decisions

Setting	Initial asset mix				Initial asset level
	Cash	Stocks	Property	Bonds	
3	0	100	0	0	42,083
3a	0	100	0	0	42,083
3b	0	100	0	0	39,811

As can be verified from Table 23, the initial decisions are only marginally affected by these changes. This indicates that the model is robust with respect to the changes in exogenous data, that are specified above.

Results from the ALM Model in Comparison with Results from the Forward-Only Procedure

When we compare the outcomes from the forward-only procedure to the results of the ALM model, it is clear that the backward procedure affects the results in a substantial way. The graphs of the development of the average funding level and the probabilities of underfunding reflect the main difference between the two solutions. The forward-only procedure apparently leads to decisions that reduce regular contributions to the maximum extent. And indeed, the present value of the contributions is lower than that of the ALM

model. However, this is achieved at cost of rendering infeasible strategies in all settings: the probabilities of underfunding are much higher than the maximum of 5% per annum. The ALM model does much better: for setting 2 and for setting 3, feasible solutions have been presented. Because no feasible solution could be found for setting 1, the objective has been changed into minimising violations of the solvency constraints. The policy that has been determined by the ALM model resulted in an average probability of underfunding equal to 11% whereas the average probability of underfunding under the forward-only policy amounted to 15%. Given these observations, it can be stated that the results of the policy from the ALM model are superior to those that have been obtained from the forward-only model. This is due to the employment of the backward procedure in conjunction with the forward procedure.

Results from The ALM Model in Comparison with Results from Optimal Static Decisions

Neither the ALM model nor the static model has resulted in a feasible policy for setting 1. The extent to which solvency constraints are violated, however, differs greatly. Whereas the policy from the ALM model results in an average probability of underfunding of 11%, that of the static policy amounts to 22%. Even more dramatic is the difference between present values of the expected costs of remedial contributions: 24,340 mln Dfl. for the static model, compared to 699 mln Dfl. for the ALM model. Notice that the difference between the present values of the expected total costs is only marginal. Again, the results from the static model suffer from the fact that the static decisions reflect a trade-off of their consequences in all states of the world.

The ALM model has determined a feasible policy for setting 2 as well as for setting 3. Let us ignore the small infeasibilities of the static policy in setting 2. Instead, compare the present values of the expected costs of funding of the two policies in settings 2 and 3. In both settings, the present value of the expected costs of funding, associated with the policy from the ALM model is the smaller one, which shows the superiority of the ALM model over the static model.

However, there is another noteworthy observation to be made. Consider the composition of the costs of total contributions. In all settings, the remedial contributions of the static model are a multiple order of magnitude higher than those of the ALM model: they are respectively 30 times, 20 times and 60 times as high in setting 1, 2 and 3. As a consequence, the level of total contributions to the fund that follows from the static model will vary enormously from one year to the next.

Finally, notice that the initial asset mixes and, in setting 3, the initial asset level from the

ALM model are rather different from those of the static model. Thus, one cannot expect to achieve results that are comparable to those from the ALM model by following a static policy and making recourse decisions when times goes by.

7.4 Summary

In this chapter, computational results have been presented that provide the following insights with respect to the ALM approach that has been proposed in this monograph:

1. Dynamic ALM strategies lead to current decisions that are different from static policies. This applies to state dependent as well as to time dependent static policies.
2. In comparison to the static models, the employment of the ALM model has resulted in strategies of which the costs of funding are lower, the probabilities of underfunding are substantially smaller and the magnitude of deficits, reflected by the costs of remedial contributions, has been reduced dramatically.
3. The favourable outcome of the comparison of policies determined by the ALM model with policies determined by optimal static decisions, are to a major extent due to:
 - the fact that probabilities of underfunding at intertemporal points in time as well as at the planning horizon are endogenous to the model and have been modelled explicitly, and
 - the dynamic character of the ALM model which enables the policies to react to situations that have emerged at the time of decision making and to reflect a correct trade-off between their longer term consequences and their short term effects.

Limited though this computational experience might be, it does tend to validate our approach towards asset liability management in a satisfactory manner.

1. The first part of the document discusses the importance of maintaining accurate records of all transactions and activities. It emphasizes that proper record-keeping is essential for transparency and accountability, particularly in financial matters. The text suggests that organizations should implement robust systems to capture and store data, ensuring that all relevant information is accessible and up-to-date.

2. The second part of the document addresses the challenges of data management and security. It highlights the need for strong security protocols to protect sensitive information from unauthorized access and cyber threats. The author recommends regular security audits and updates to software and hardware to mitigate risks. Additionally, the text discusses the importance of employee training and awareness in maintaining data security.

3. The third part of the document focuses on the integration of technology into business processes. It argues that leveraging modern tools and platforms can significantly improve efficiency and productivity. The text provides examples of various software solutions, such as project management tools, CRM systems, and automation software, and discusses how they can be effectively implemented within an organization.

4. The fourth part of the document explores the role of leadership in driving organizational success. It stresses that effective leaders must be able to inspire and motivate their teams, set clear goals, and foster a culture of innovation and collaboration. The text offers practical advice on how leaders can develop their skills and build strong relationships with their employees.

5. The fifth part of the document discusses the importance of continuous learning and development. It suggests that organizations should invest in training and development programs to help employees stay current in their fields and acquire new skills. The text also emphasizes the value of mentorship and peer learning in promoting growth and innovation within the organization.

6. The sixth part of the document addresses the issue of sustainability and social responsibility. It argues that organizations have a responsibility to their stakeholders beyond just providing products and services. The text discusses the benefits of sustainable practices, such as reducing environmental impact and promoting social equity, and provides guidance on how organizations can integrate these values into their business strategies.

7. The seventh part of the document discusses the importance of communication and collaboration. It suggests that effective communication is key to ensuring that all team members are aligned and working towards common goals. The text provides tips on how to improve communication skills, such as active listening and clear communication, and emphasizes the importance of fostering a collaborative work environment.

8. The eighth part of the document discusses the importance of innovation and creativity. It argues that organizations must be able to think outside the box and develop new ideas to stay competitive in a rapidly changing market. The text provides strategies for encouraging innovation, such as creating a culture of experimentation and providing resources for research and development.

9. The ninth part of the document discusses the importance of flexibility and adaptability. It suggests that organizations must be able to respond quickly to changes in the market and adjust their strategies accordingly. The text provides advice on how to build a flexible organization, such as having a clear vision and mission statement and being open to new ideas and perspectives.

10. The tenth part of the document discusses the importance of resilience and perseverance. It argues that organizations must be able to withstand challenges and setbacks and remain focused on their long-term goals. The text provides strategies for building resilience, such as having a strong support system and maintaining a positive attitude.

Appendix A. Chance Constrained Programming, Normally Distributed Returns and Mean-Variance Optimisation

In chapter 2 the relationship between mean-variance surplus optimisation as formulated in *MV* and chance constraint programming under the assumption of normally distributed returns as reflected by *CC* was discussed. It was claimed that any optimal solutions to one model can also be obtained as an optimal solution to the other model. To formulate the relationship between these models more precisely:

$X, E[B_T], \sigma^2[B_T]$ with $E[B_T] > 0$ is an optimal solution to *MV* if and only if

$X, E[B_T], \sigma^2[B_T]$ is an optimal solution to *CC* with $\psi^u = \varphi\left(\frac{E[B_T]}{\sigma[B_T]}\right)$.

Proof: Since the two models are identical with the exception of equations (8),(9) and the upper bound on $\sigma^2[B_T]$, it suffices to show that $X, E[B_T], \sigma^2[B_T]$ with $E[B_T] > 0$ satisfies $\sigma^2[B_T] \leq \sigma^2[B_T]^u$ if and only if $X, E[B_T], \sigma^2[B_T]$ satisfies (8) and (9) with

$\psi^{up} = \varphi\left(\frac{E[B_T]}{\sigma[B_T]}\right)$. $X, E[B_T], \sigma^2[B_T]$ satisfies (8) and (9) with $\psi^u = \varphi\left(\frac{E[B_T]}{\sigma[B_T]}\right)$ iff

$$E[B_T] - \varphi^{-1}\left(\varphi\left(\frac{E[B_T]}{\sigma[B_T]^u}\right)\right) \sigma[B_T] \geq 0 \Leftrightarrow$$

$$E[B_T] - E[B_T] \frac{\sigma[B_T]}{\sigma[B_T]^u} \geq 0 \Leftrightarrow$$

$\sigma^2[B_T] \leq \sigma^2[B_T]^u$ since $E[B_T] > 0$. So $X, E[B_T], \sigma^2[B_T]$ with $E[B_T] > 0$ satisfies the upper bound on the variance of the ultimate surplus if and only if $X, E[B_T], \sigma^2[B_T]$

satisfies (8) and (9) with $\psi^u = \varphi\left(\frac{E[B_T]}{\sigma[B_T]}\right)$ \square

Appendix B. The ALM Model With Transaction Costs

In order to extend the ALM model with transaction costs, these costs have to be accounted for in two respects: it has to be registered how much costs are incurred from trading and this has to be reflected in the constraint that reflects the asset level at the beginning of the year.

In this appendix the following additional notation will be used:

c_i costs of buying asset class i , as a fraction of the amount to be bought,

d_i costs of selling asset class i , as a fraction of the amount to be sold,

To register the amounts that are bought and sold as of $t=1$, it suffices to include the following equation in the ALM model:

$$\text{For } t=1,\dots,T-1, s=1,\dots,S_{T-1}, i=1,\dots,N$$

$$X_{its} = h_{its} X_{i,t-1,s} + (1-d_i) D_{its} - C_{its} \quad (184)$$

To reflect the effect of transaction costs on the asset level at the beginning of each year, (35) has to be replaced by:

$$\text{For } t=1,\dots,T-1, s=1,\dots,S_{T-1}$$

$$A_{ts} + Y_{ts} - l_{ts} - \sum_{i=1}^N (d_i D_{its} + c_i C_{its}) = \sum_{i=1}^N X_{its} \quad (185)$$

If one chooses to account for transaction costs at time 0 as well, then, analogous equations have to be included, given the asset mix at time 0, before the first allocation decision has been made.

Appendix C. Illustrations of Importance Sampling

Given a bowl with 5 balls, each of which carries a number, one is allowed to draw 3 times with replacement. Then one is asked to estimate the mean of the numbers that are printed on the balls in the bowl. Suppose that, after drawing 3 times, the numbers 1, 2 and 15 have been observed. The classical unbiased estimate of the mean of the 5 numbers in the bowl would be the sample mean: $1/3(1+2+15) = 6$.

Now, let us assume that there is additional information: there are two balls with a number smaller than 4 and 3 balls which carry numbers that exceed 12. One would hope to sample balls that carry the larger numbers: they contribute most to the mean that has to be estimated, so once they are observed, the remaining uncertainty is relatively small.

Each of the larger numbers is greater than 4 times any of the smaller numbers. Therefore, a new distribution, which will be referred to as the importance sampling distribution, can be specified of which it may be expected that the probability of drawing a number is proportional to its relative contribution to the mean of the population, or at least, the importance sampling distribution will come closer to this property than the original distribution in which all numbers are equally likely to be drawn.

Suppose that the sampling process has been manipulated in such a way that the probability of drawing any of the smaller numbers equals $1/14$ and the probability of drawing any of the larger numbers is $4/14$, and let the numbers 2, 16 and 20 be drawn. How should the mean of the population be estimated, given these observations, the original probability distribution and the importance sampling distribution? Since the sample has been manipulated, it would be incorrect to estimate the mean of the population by the sample average. Instead, one should construct a correction factor for each of the observations, so that they contribute to the extent that is in agreement with the original probability distribution. Because the sample has been drawn from the importance sampling distribution, the correction factor of the balls with the larger numbers should be $\frac{1/5}{4/14} = 0.7$ and

the correction factor for the balls which carry the smaller numbers should be

$\frac{1/5}{1/14} = 2.8$. It follows that the mean should now be estimated as $1/3(2.8 * 2 + 0.7 * 16 + 0.7 * 20) = 10.27$. It will be shown in ? that both estimates of the mean are unbiased. Nevertheless, the difference is more than 70% of the smaller estimate.

In order to analyze the variance of each estimator, suppose that the bowl contained balls

that are numbered 1, 2, 15, 16 and 20. Then, under the original probability distribution, the mean and variance of the population are 10.8 and 60.56 respectively. It follows that the variance of the first estimator equals $1/9(60.56 + 60.56 + 60.56) = 20.19$.

To analyze the variance of the importance sampling estimator, notice that drawing numbers and then multiplying them by a correction factor is equivalent to multiplying the numbers first and sampling from the population $2.8 * 1$, $2.8 * 2$, $0.7 * 15$, $0.7 * 16$, $0.7 * 20$, where each of the first two elements has probability $1/14$ to be sampled and each of the other elements is drawn with probability $4/14$. The mean of the importance sampling distribution is equal to $1/14(2.8 + 5.6) + 4/14(10.5 + 11.2 + 14) = 10.8$. Apparently, the correction factors have indeed been chosen in such a way that the mean of importance sampling distribution is equal to the mean of the original population. The variance of the importance sampling distribution is equal to $1/14((2.8 - 10.8)^2 + (5.6 - 10.8)^2) + 4/14((10.5 - 10.8)^2 + (11.2 - 10.8)^2 + (14 - 10.8)^2) = 9.5$. As a consequence, the variance of the importance sampling estimator of the mean of the original distribution equals $1/9(9.5 + 9.5 + 9.5) = 3.17$. I.e., the variance of the estimator of the mean has been reduced by more than 80%.

This example shows that additional information on the distribution of a random variable can be exploited to specify an importance sampling distribution which in turn enables one to obtain an estimate of the mean of the unknown population that is superior to the average of a sample from the original distribution.

As has been shown in (166), the optimal choice of the importance sampling distribution reduces the variance of the estimate to zero. To illustrate that case as well, consider Table 24. The second and third column present the importance sampling probability and the associated correction factor. As in the previous example, drawing from the importance sampling distribution and multiplying the observations by the correction factor is equivalent to multiplying all elements of the original population by the associated correction factor and sampling from the original density.

Table 24. Optimal importance sampling probabilities

number	importance sampling probability	correction factor	number * correction factor
1	$1/(1+2+15+16+20) = 1/54$	$\frac{1/5}{1/54}$	10.8
2	$2/54$	$\frac{1/5}{2/54}$	10.8
15	$15/54$	$\frac{1/5}{15/54}$	10.8
16	$16/54$	$\frac{1/5}{16/54}$	10.8
20	$20/54$	$\frac{1/5}{20/54}$	10.8

As can be verified from the fourth column, every observation from such a sample yields the same number: 10.8, the mean of the original population. It follows that any importance sample of any size will result in an estimated mean equal to 10.8 with zero variance.

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Samenvatting (Summary in Dutch)

In dit proefschrift wordt een optimalisatiemodel gepresenteerd voor het analyseren van beleggingsbeleid en premiebeleid van pensioenfondsen, rekening houdend met de ontwikkeling van verplichtingen in samenhang met economische omstandigheden. Een dergelijk beleid heet **asset liability management (ALM)**. De aanpak die hier wordt voorgesteld is gebaseerd op scenario analyse.

Met deze aanpak kan een dynamisch ALM beleid bepaald worden, zodanig dat de kosten om een gegeven pensioenreglement te financieren geminimaliseerd worden onder de voorwaarden dat de kans op onderdekking acceptabel klein is en de pensioenpremies van jaar op jaar voldoende stabiel zijn.

Probleembeschrijving

Pensioenfondsen hebben de taak om pensioenen uit te betalen aan deelnemers aan het pensioenfonds die hun actieve carrière hebben beëindigd en aan nabestaanden daarvan. Deze uitkeringen dienen in overeenstemming te zijn met het pensioenreglement waarin is vastgelegd op welke wijze de hoogte van de uitkering berekend wordt, wanneer deze ingaat en wanneer deze beëindigd wordt.

Gedurende de tijd neemt het vermogen van het pensioenfonds, de waarde van beleggingsportefeuille, toe met beleggingsopbrengsten en premieontvangsten. Het neemt af door het uitbetalen van pensioenen. Het is de taak van het pensioenfonds om deze inkomende en uitgaande geldstromen in evenwicht te houden zodat het fonds aan de geldende solvabiliteitseisen voldoet en dat alle uitkeringen, nu en in de toekomst, tijdig gedaan kunnen worden.

Beslissingen die in belangrijke mate bepalen of het pensioenfonds erin slaagt haar taak uit te voeren zijn de **hoogte van de pensioenpremies** en de verdeling van te beleggen gelden over de beleggingscategorieën waarin het pensioenfonds wil beleggen. Deze verdeling wordt de **beleggingsmix** genoemd.

De hoogte van de pensioenpremie dient zodanig vastgesteld te worden dat de sponsor in staat en bereid is de premies te betalen. Deze beperking komt vaak tot uiting in een bovengrens op de pensioenpremie en op eventuele premieverhogingen, beide als percentage van de loonkosten.

In principe is het pensioenfonds vrij in haar keuze van een beleggingsmix. De facto leiden

algemeen geaccepteerde noties van acceptabele en onacceptabele beleggingsmixen echter al snel tot minima en maxima op het aandeel van verschillende beleggingscategorieën in de beleggingsmix. Bij het effectueren van veranderingen in de beleggingsmix dient bovendien rekening gehouden te worden met de liquiditeit van de effectenmarkten waarin men actief is.

Onzekerheden, waardering van verplichtingen en beleggingen

In het pensioenreglement is vastgelegd op welke wijze de hoogte van toekomstige uitkeringen berekend dient te worden. De **hoogte van de toekomstige uitkeringen** is echter onzeker. Deze is onder meer afhankelijk van toekomstige ontwikkelingen in het deelnemersbestand, zoals sterfte, invaliditeit, loopbaanontwikkeling en burgerlijke staat. Voor veel Nederlandse pensioenfondsen is de belangrijkste bron van onzekerheid de toekomstige ontwikkeling van prijsinflatie en toekomstige loonronden: de hoogte van oudedagspensioenen bedraagt veelal zeventig procent van het laatst verdiende loon²⁸. Gevolg hiervan is dat pensioenrechten van actieve deelnemers die zijn opgebouwd tijdens eerdere dienstjaren toenemen met looninflatie. Pensioenrechten van slapers en ingegane pensioenen zijn vaak geïndexeerd met prijsinflatie.

In de regel doen pensioenfondsen slechts conditionele indexatietoezeggingen, dat wil zeggen dat indexatie slechts plaatsvindt voor zover de financiële situatie van het pensioenfonds dat toelaat. In het vervolg zullen we echter geen onderscheid maken naar conditionele en onconditionele verplichtingen. Als een reglement conditionele indexatietoezeggingen bevat, dan is onze ALM aanpak erop gericht geïndexeerde pensioenen uit te keren.

ALM beleid

Goed ALM beleid bestaat uit beleggingsbeslissingen en premiebeleid die resulteren in een gewenste risico-rendement structuur in termen van de ontwikkeling van de financiële staat van het pensioenfonds. Daarbij worden de pensioenkosten geminimaliseerd terwijl gezorgd wordt dat het fonds aan haar verplichtingen kan voldoen. Helaas kan zelfs bij een perfecte uitvoering van een uitmuntende ALM strategie niet gegarandeerd worden dat alle toezeggingen onder alle omstandigheden kunnen worden nagekomen.

²⁸ Inclusief een staatspensioen op basis van de Algemene Ouderdoms Wet (AOW). Het AOW bestanddeel is wel geïndexeerd, maar onafhankelijk van de salarisontwikkeling. Nu de hoogte van de AOW uitkering onderwerp van politieke discussie is geworden, lopen pensioenfondsen het risico dat zij hogere uitkeringen zullen moeten doen ter compensatie van eventuele lagere AOW uitkeringen.

Indien de verplichtingen geïndexeerd zijn met inflatie, bij voorbeeld, kan het in tijden van uitzonderlijk hoge inflatie voorkomen dat het fonds niet aan alle verplichtingen kan voldoen zonder extreem hoge premies te heffen. De kans bestaat dat pensioenfondsen in een situatie verzeild raken waarin de waarde van de beleggingen kleiner is dan die van de verplichtingen. Deze kans noemen we de **kans op onderdekking**. Het ALM beleid moet er dan ook op gericht zijn de kans op onderdekking acceptabel klein te houden.

Noch beleggingsmixen, noch premiehoogtes worden vastgesteld voor de gehele planningsperiode. Beslissingen worden bijgesteld als veranderende omstandigheden, zoals een gewijzigde dekkingsgraad en een gewijzigde kijk op de toekomstige ontwikkeling van de economische omstandigheden, daartoe nopen. Stabiliteitseisen aan het ALM beleid, zoals een maximum op premieverhogingen, kunnen echter impliceren dat corrigerende beslissingen slechts in beperkte mate genomen kunnen worden. Hieruit volgt dat **huidige beslissingen en toekomstige beslissingen niet onafhankelijk** van elkaar genomen kunnen worden. Daarom dient een ALM beleid te bestaan uit beslissingen die nu genomen worden en uit reeksen van beslissingen die in de toekomst genomen worden. Toekomstige beslissingen dienen afhankelijk te zijn van situatie die ontstaan is op het moment dat de beslissing genomen wordt. Huidige beslissingen dienen te anticiperen op de mogelijkheid om beslissingen later bij te sturen. Voor zover zij de toekomstige beslissingsvrijheid beperken dienen zij bovendien het resultaat te zijn van een correcte afweging van korte termijn effecten en langere termijn effecten. Zo'n beleid wordt een **dynamische politiek** genoemd.

Scenario's om een onzekere toekomst te modelleren

Een van de cruciale aspecten van een ALM model is de wijze waarop onzekerheid gemodelleerd wordt. Wij modelleren onzekerheid met behulp van een groot aantal scenario's. Elk van deze scenario's bestaat uit een reeks van opeenvolgende toestanden en geeft een mogelijk verloop weer van de omgeving waarin ALM beslissingen genomen dienen te worden. **Toekomstige omstandigheden** worden gedefinieerd door de hoogte van de loonsom, de hoogte van de verplichtingen, de hoogte van de uitkeringen en het totale rendement op elk van de beleggingscategorieën over de voorafgaande periode. Deze mogelijke toestanden zijn onafhankelijk van de te nemen beslissingen met betrekking tot de beleggingsmix en het premiebeleid. Ze worden volledig gedefinieerd door factoren die exogeen zijn aan het beslissingsmodel.

De scenario's worden zo gegenereerd dat toekomstige toestanden van de wereld consistent zijn, dat wil zeggen dat deterministische en stochastische relaties tussen toestandsvariabelen erin tot uitdrukking komen. Dat geldt zowel voor relaties tussen

toestandsvariabelen die tezamen één toestand definiëren als voor relaties tussen toestandsvariabelen van opeenvolgende toestanden. Beleidsmakers kunnen zelf aangeven welke mogelijke toekomstige ontwikkelingen zij bij het bepalen van hun ALM beleid relevant achten. De scenariogenerator kan dan gebruikt worden om alle toekomstige omstandigheden te weerspiegelen waarvan de beslisser vindt dat er bij de besluitvorming rekening mee gehouden moet worden.

In dit proefschrift zijn de scenario's gebaseerd op een tijdreeksmodel. De ALM methode die hier wordt voorgesteld kan echter ook gebruikt worden in combinatie met scenario's die op een andere manier gegenereerd zijn. Zo kan men er bij voorbeeld voor kiezen om bij de scenariogeneratie een model te gebruiken dat gebaseerd is op economische theorie.

Een dynamisch optimalisatiemodel voor asset liability management

In hoofdstuk 3 wordt een optimalisatiemodel gepresenteerd waarmee een ALM beleid bepaald kan worden dat voor elke toestand de premiehoogte en de beleggingsmix specificeert. Deze beslissingen bepalen eveneens voor elke toestand de waarde van de beleggingsportefeuille en, in combinatie met de exogeen bepaalde waarden van de verplichtingen, de dekkingsgraad. De beslissingen in alle toestanden worden simultaan bepaald. Dit maakt het mogelijk om zowel afwegingen te maken tussen korte termijn effecten en langere termijn effecten als afwegingen tussen de consequenties van beslissingen in verschillende mogelijke toekomstige toestanden. Het model is ontwikkeld ter ondersteuning van het formuleren van ALM beleid dat:

- een acceptabel kleine kans op onderdekking waarborgt,
- voldoende stabiele premies als percentage van de loonsom garandeert,
- de pensioenkosten, gekwantificeerd als de contante waarde van de verwachte pensioenpremies, minimaliseert.

De kans op onderdekking is zodanig gemodelleerd dat:

1. het model gebruikt kan worden met **elke kansverdeling** die door scenario's kan worden weergegeven. Dus ook met kansverdelingen die impliciet gespecificeerd zijn. Een voorbeeld hiervan is de kansverdeling van de ontwikkeling van verplichtingen. Deze is vaak impliciet gegeven door het computerprogramma waarmee de uit te keren bedragen op basis van het pensioenreglement, loonronden en prijsinflatie berekend worden,
2. vanuit elke toestand de kans dat één jaar later een situatie van onderdekking optreedt,

acceptabel klein is,

3. bij de optimalisatie **expliciet** rekening wordt gehouden met kansen op onderdekking. Zowel op het eind van de **planningsperiode** als **tussentijdstippen**.

Samenvattend, in deze dissertatie wordt een **gemengd geheeltallig stochastisch programmerings model** voorgesteld. Het kan gebruikt worden om **dynamische ALM strategieën** te bepalen die gebaseerd zijn op scenario's die elke set van vooronderstellingen kunnen weerspiegelen waarop men het ALM beleid wenst te baseren. **Kansen op onderdekking** worden daarbij **expliciet** gemodelleerd op basis van **realistische vooronderstellingen** ten aanzien van de kansverdelingen van exogene kansvariabelen.

Voor zover ons bekend is dit model het eerste waarmee een dynamisch beleid bepaald kan worden dat gebaseerd wordt op realistische vooronderstellingen.

Rekenkundige experimenten

Teneinde inzicht te krijgen in het gedrag van het model bij toepassing op realistische problemen, zijn er ALM politieken berekend op basis van de data van een Nederlands pensioenfonds met een actuariële reserve van ruim 16 miljard gulden en ongeveer 1 miljoen participanten, waarvan ca. 240.000 actieven.

De rekenresultaten zijn verkregen door de heuristiek toe te passen die wordt gepresenteerd in hoofdstuk 5. Het meer-perioden beslissingsprobleem wordt daarbij benaderd door een reeks van twee-perioden problemen. De benodigde rekentijd om met deze heuristiek een dynamisch ALM beleid te bepalen is proportioneel met het aantal perioden dat in beschouwing wordt genomen. De resultaten geven de volgende indicaties met betrekking tot de ALM methode die in dit proefschrift is gepresenteerd:

1. Dynamische ALM strategieën leiden tot huidige beslissingen die afwijken van beslissingen die genomen worden binnen een statische politiek.
2. In vergelijking met statische modellen resulteert het gebruik van dynamische modellen in lagere pensioenkosten, kleinere kansen op onderdekking en, in geval van onderdekking, een enorme reductie van de mate van onderdekking,
3. De bevredigende uitkomst van de vergelijking van ALM beleid, bepaald door het voorgestelde dynamische model, en beleid, bepaald door statische modellen, zijn voor het

grootste deel toe te schrijven aan:

- het feit dat kansen op onderdekking op zowel op tussentijdstippen als op de planningshorizon expliciet zijn gemodelleerd en endogeen zijn aan het beslissingsmodel en
- het dynamische karakter van het ALM model dat de mogelijkheid biedt om te reageren op gewijzigde omstandigheden en om een correcte afweging te maken tussen kortere termijn en langere termijn effecten.







