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Von Mises-Type Conditions in Second Order Regular Variation

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We give a thorough treatment concerning sufficient conditions involving derivatives for extended regular variation of second order. Most of the results are new. A summary of the analogous (known) results for first order extended regular variation is given first.

1. VON MISES-TYPE CONDITIONS IN EXTENDED REGULAR VARIATION

DEFINITION. A function f satisfies the extended regular variation condition if there exists a positive function a such that for all $x > 0$,

$$\lim_{t \to \infty} \frac{f(tx) - f(t)}{a(t)} = \frac{x^{\gamma} - 1}{\gamma}, \tag{1.1}$$

where for $\gamma = 0$ the right-hand side is interpreted as $\log x$.

We summarize some results concerning this class of functions.

Property 1. Suppose f satisfies (1.1).

a. For $\gamma \geq 0$,

$$\lim_{t \to \infty} \frac{a(t)}{f(t)} = \gamma. \tag{1.2}$$

Hence for all $x > 0$,

$$\lim_{t \to \infty} \frac{f(tx)}{f(t)} = x^{\gamma}. \tag{1.3}$$

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b. For $\gamma < 0$,

$$f(\infty) := \lim_{t \to \infty} f(t) \tag{1.4}$$

exists and

$$\lim_{t \to \infty} \frac{a(t)}{f(\infty) - f(t)} = \gamma. \tag{1.5}$$

Hence for all $x > 0$,

$$\lim_{t \to \infty} \frac{f(\infty) - f(tx)}{f(\infty) - f(t)} = x^{\gamma}. \tag{1.6}$$

Relations (1.5) and (1.6) also hold if $\gamma = 0$ and $f(\infty)$ exists. (See, e.g., Bingham *et al*. [2, Section 3.2] and Geluk and de Haan [6, Theorem 1.10].)

Remark. Note that for $\gamma > 0$ relation (1.3) implies (1.1) and for $\gamma < 0$ relation (1.6) implies (1.1). A function f satisfying (1.3) is said to be *regularly varying* (or of *regular variation*) with *index* γ (notation $f \in \mathrm{RV}_\gamma$).

Property 2.

a. Suppose f is differentiable and for all $x > 0$,

$$\lim_{t \to \infty} \frac{f'(tx)}{f'(t)} = x^{\gamma - 1}, \tag{1.7}$$

then (1.1) holds with $a(t) = tf'(t)$.

b. Conversely, if f satisfies (1.1), f is differentiable, and f' is monotone, then (1.7) holds. (See e.g., de Haan [3, pp. 13 and 21] and Bingham *et al*. [2, Section 1.7.3].)

Property 3. Suppose f satisfies (1.1). Then there exists a twice differentiable function f_1 with

$$f(t) - f_1(t) = o(a(t)) \quad (t \to \infty) \tag{1.8}$$

and such that

$$\lim_{t \to \infty} \frac{tf_1''(t)}{f_1'(t)} = \gamma - 1. \tag{1.9}$$

Remark. Conversely, (1.8) and (1.9) imply (1.1). (See e.g., [2, Sections 1.8 and 3.7] and [6, Corollaries 2.12 and 2.16].)

Property 4. Suppose f satisfies (1.1). Let f_1 be a function with the property that for all $a > 1$ there exists t_0 such that for $t \geq t_0$,

$$f(t/a) \leq f_1(t) \leq f(ta). \tag{1.10}$$

Then

$$f(t) - f_1(t) = o(a(t))$$

and hence f_1 satisfies (1.1). (See e.g. [6, Proposition 1.22] "inversely asymptotic.")

2. VON MISES-TYPE CONDITIONS IN EXTENDED REGULAR VARIATION OF SECOND ORDER

We are going to prove results analoguous to Properties 1–4 for extended regular variation of second order. A function f is of extended regular variation of second order if for all $x > 0$,

$$\lim_{t\to\infty} \frac{f(tx) - f(t) - a(t)(x^\gamma - 1)/\gamma}{c(t)} = \int_1^x s^{\gamma-1} \int_1^s u^{\rho-1}\, du\, ds \quad (2.1)$$

for some function a (assumed positive) and c (which is necessarily of constant sign eventually). Here $\gamma \in \mathbb{R}$ and $\rho \leq 0$ are parameters. See de Haan and Stadtmüller [5]. We shall need the following properties of the function a and c:

$$a \in \mathrm{RV}_\gamma, \quad (2.2)$$

$$c \in \mathrm{RV}_{\rho+\gamma}, \quad (2.3)$$

$$\lim_{t\to\infty} \frac{a(tx) - x^\gamma a(t)}{c(t)} = x^\gamma \frac{x^\rho - 1}{\rho}, \quad (2.4)$$

hence (since $\rho \leq 0$)

$$\lim_{t\to\infty} \frac{c(t)}{a(t)} = 0. \quad (2.5)$$

THEOREM 1. *Suppose that f is twice differentiable and f' positive. Write*

$$A(t) := \frac{tf''(t)}{f'(t)} - \gamma + 1.$$

a. *Suppose*

$$\mathrm{sgn}(A(t)) \text{ is constant for large } t, \quad (2.6)$$

$$\lim_{t\to\infty} A(t) = 0 \quad (2.7)$$

and

$$|A| \in \mathrm{RV}_\rho \qquad \text{for some } \rho \leq 0. \tag{2.8}$$

Then

$$\lim_{t\to\infty}\left(\frac{f(tx)-f(t)}{tf'(t)}-\frac{x^\gamma-1}{\gamma}\right)\Bigg/\left\{\frac{tf''(t)}{f'(t)}-\gamma+1\right\}$$
$$=\int_1^x s^{\gamma-1}\int_1^s u^{\rho-1}\,du\,ds. \tag{2.9}$$

(b). *Conversely, suppose*

$$\lim_{t\to\infty}\frac{f(tx)-f(t)-a(t)(x^\gamma-1)/\gamma}{c(t)}=\int_1^x s^{\gamma-1}\int_1^s u^{\rho-1}\,du\,ds =: H(x) \tag{2.10}$$

for some $a>0$ *and* c *of constant sign. If* A *is eventually monotone, then* (2.6), (2.7), *and* (2.8) *hold.*

Proof. (a) See de Haan and Resnick [4, Theorem 2.1].

(b) Suppose (2.10) holds with a positive function c (a similar proof applies for negative c). Since the derivative of $\log f'(t)-(\gamma-1)\log t$ is $t^{-1}A(t)$ and since A has constant sign, we find that $t^{-\gamma+1}f'(t)$ is eventually monotone. Suppose $t^{-\gamma+1}f'(t)$ is non-decreasing (if non-increasing, a similar proof applies). Now

$$\int_1^x\frac{(ts)^{-\gamma+1}f'(ts)-t^{-\gamma}a(t)}{t^{-\gamma}c(t)}s^{\gamma-1}\,ds$$
$$=\frac{f(tx)-f(t)-a(t)(x^\gamma-1)/\gamma}{c(t)}\to H(x)$$

($t\to\infty$). For $x>1$ the left-hand side is bounded below by

$$\frac{t^{-\gamma+1}f'(t)-t^{-\gamma}a(t)}{t^{-\gamma}c(t)}\int_1^x s^{\gamma-1}\,ds.$$

Hence

$$\lim_{x\downarrow 1}\limsup_{t\to\infty}\frac{t^{-\gamma+1}f'(t)-t^{-\gamma}a(t)}{t^{-\gamma}c(t)}\leq\lim_{x\downarrow 1}H(x)\Bigg/\int_1^x s^{\gamma-1}\,ds=0.$$

The corresponding inequality for lim inf follows by taking $0 < x < 1$ and letting $x \uparrow 1$. It follows that

$$\lim_{t\to\infty} \frac{tf'(t) - a(t)}{c(t)} = 0. \tag{2.11}$$

Hence (cf. (2.10)) we have

$$\lim_{t\to\infty} \frac{f(tx) - f(t) - tf'(t)(x^\gamma - 1)/\gamma}{c(t)} = H(x).$$

We also know now that $f' \in \mathrm{RV}_{\gamma-1}$.

Next we continue in the same fashion:

$$\frac{f(tx) - f(t) - tf'(t)(x^\gamma - 1)/\gamma}{c(t)}$$

$$= \int_1^x \frac{tf'(ts) - tf'(t)s^{\gamma-1}}{c(t)}\, ds$$

$$= \int_1^x \frac{(ts)^{1-\gamma} f'(ts) - t^{1-\gamma} f'(t)}{c(t)t^{-\gamma}}\, s^{\gamma-1}\, ds$$

$$= \int_1^x s^{\gamma-1} \int_1^s \frac{t\left\{(tu)^{1-\gamma} f''(tu) + (1-\gamma)(tu)^{-\gamma} f'(tu)\right\}}{c(t)t^{-\gamma}}\, du\, ds$$

$$= \int_1^x s^{\gamma-1} \int_1^s u^{-\gamma} \frac{tuf''(tu)/f'(tu) - \gamma + 1}{c(t)/(tf'(t))} \frac{f'(tu)}{f'(t)}\, du\, ds$$

$$= \int_1^x s^{\gamma-1} \int_1^s u^{-\gamma} \frac{A(tu)}{c(t)/(tf'(t))} \frac{f'(tu)}{f'(t)}\, du\, ds.$$

Since the left-hand side converges to $H(x)$, A is monotone, $f' \in \mathrm{RV}_{\gamma-1}$ and $c(t) \in \mathrm{RV}_{\rho+\gamma}$, we find as before

$$\lim_{t\to\infty} \frac{A(t)}{c(t)/(tf'(t))} = 1,$$

hence $|A| \in \mathrm{RV}_\rho$. Since c is of constant sign and since $c(t) = o(a(t))$ $(t \to \infty)$ in (2.10), we also find that A has constant sign and $\lim_{t\to\infty} A(t) = 0$.

Remark. Note that (2.4) and (2.11) imply

$$\lim_{t\to\infty} \frac{f'(tx)/f'(t) - x^{\gamma-1}}{c(t)/(tf'(t))} = x^{\gamma-1} \frac{x^\gamma - 1}{\gamma}$$

for $x > 0$.

Remark. For regularly varying functions f the following simpler result holds.

Suppose f is differentiable. Write $B(t) := tf'(t)/f(t) - \gamma$.

(a). Suppose

$$\operatorname{sgn}(B(t)) \text{ is constant for large } t. \tag{2.12}$$

$$\lim_{t\to\infty} B(t) = 0 \tag{2.13}$$

and

$$|B| \in \mathrm{RV}_\rho \qquad \text{for some } \rho \le 0. \tag{2.14}$$

Then

$$\lim_{t\to\infty} \frac{f(tx)/f(t) - x^\gamma}{q(t)} = x^\gamma \int_1^x u^{\rho-1}\,du \qquad \text{for all } x > 0$$

holds for $q = B$.

(b). Conversely, suppose (2.15) holds for some function $q \neq 0$ and B is eventually monotone, then (2.12), (2.13), and (2.14) hold.

THEOREM 2. *Suppose*

$$\lim_{t\to\infty} \frac{f(tx) - f(t) - a(t)(x^\gamma - 1)/\gamma}{c(t)} = H(x). \tag{2.16}$$

Then there exists a twice differentiable function f_1 with

$$\lim_{t\to\infty} (f(t) - f_1(t))/c(t) = 0, \tag{2.17}$$

$$\lim_{t\to\infty} (a(t) - tf_1'(t))/c(t) = 0, \tag{2.18}$$

$$\lim_{t\to\infty} a(t)A_1(t)/c(t) = 1 \tag{2.19}$$

and such that, *with*

$$A_1(t) := \frac{tf_1''(t)}{f_1'(t)} - \gamma + 1, \tag{2.20}$$

$$\operatorname{sgn}(A_1(t)) \text{ is constant eventually}, \tag{2.21}$$

$$\lim_{t\to\infty} A_1(t) = 0, \tag{2.22}$$

$$|A_1| \in \mathrm{RV}_\rho. \tag{2.23}$$

Proof. For the case $\gamma = \rho = 0$ the proof is given in [1, Appendix]. For other values of γ and ρ separate proofs apply. As an example we give the proof for $\rho = 0, \gamma > 0$. Assume that the function c is positive (for negative c a similar proof applies). Then (2.16) implies [5, Theorem 2]

$$\lim_{t\to\infty} \frac{(tx)^{-\gamma} f(tx) - t^{-\gamma} f(t)}{t^{-\gamma} c_0(t)} = \log x \tag{2.24}$$

for all $x > 0$, hence (2.16) holds with $a(t) = \gamma f(t) + \gamma^{-1} c(t)$ and $c(t) = \gamma c_0(t)$.

Now (2.24) says that the function $t^{-\gamma} f(t)$ is in the class Π, hence by Proposition 3 there is a function g_1 with

$$\lim_{t\to\infty} \frac{t^{-\gamma} f(t) - g_1(t)}{t^{-\gamma} c(t)} = \lim_{t\to\infty} \frac{f(t) - t^{\gamma} g_1(t)}{c(t)} = 0$$

and such that

$$\lim_{t\to\infty} \frac{t g_1''(t)}{g_1'(t)} = -1. \tag{2.26}$$

Combining (2.24), (2.25), and

$$\lim_{t\to\infty} \frac{g_1(tx) - g_1(t)}{t g_1'(t)} = \log x,$$

we find

$$\lim_{t\to\infty} \frac{t g_1'(t)}{t^{-\gamma} c_0(t)} = 1,$$

i.e.,

$$\lim_{t\to\infty} \frac{\gamma t^{\gamma+1} g_1'(t)}{c(t)} = 1. \tag{2.27}$$

We take $f_1(t) := t^{\gamma} g_1(t)$. Then (2.17) holds by (2.25). Further,

$$\begin{aligned}
\frac{a(t) - t f_1'(t)}{c(t)} &= \frac{\gamma f(t) + \gamma^{-1} c(t) - t f_1'(t)}{c(t)} \\
&= \gamma \frac{f(t) - t^{\gamma} g_1(t)}{c(t)} + \gamma^{-1} + \frac{\gamma t^{\gamma} g_1(t) - t(t^{\gamma} g_1(t))'}{c(t)} \\
&= \gamma \frac{f(t) - t^{\gamma} g_1(t)}{c(t)} + \gamma^{-1} - \frac{t^{\gamma+1} g_1'(t)}{c(t)} \to 0
\end{aligned}$$

$(t \to \infty)$ by (2.25) and (2.27). Hence (6) holds. Finally by (2.18), (2.26), and (2.27),

$$\begin{aligned} a(t) A_1(t) &\sim t f_1'(t) A_1(t) \\ &= t^2 f_1''(t) - (\gamma - 1) t f_1'(t) \\ &= (\gamma + 1) t^{\gamma+1} g_1'(t) + t^{\gamma+2} g_1''(t) \\ &= t^{\gamma+1} g_1'(t) \left\{ \frac{t g_1''(t)}{g_1'(t)} + \gamma + 1 \right\} \\ &\sim \gamma t^{\gamma+1} g_1'(t) \sim c(t) \qquad (t \to \infty). \end{aligned}$$

Hence (2.21), (2.22), (2.23), and (2.19) hold.

THEOREM 3. *Suppose f satisfies* (2.1) *for $\gamma = \rho = 0$ and some choice of the functions a and c. Then for all $x > 0$,*

$$\lim_{t \to \infty} \left(\left(\frac{f(tx^{1/2}) - f(tx^{-1/2})}{\log x} - a(t) \right) \Big/ c(t) \right) = 0, \tag{2.28}$$

and for all x and $y > 0$,

$$\lim_{t \to \infty} \frac{f(txy) - f(tx) - f(ty) + f(t)}{c(t)} = xy. \tag{2.29}$$

So both a and c can be expressed in a simple way as functionals of f.

Proof. Relation (2.29) has been proved by Omey and Willekens [7]. Relations (2.28) and (2.29) are easily verified.

Remark. Similar statements can be made for $\gamma \neq 0$ and/or $\rho \neq 0$, based on Theorem 2 [5]. They differ from case to case, so they are omitted.

THEOREM 4. *Suppose f satisfies* (2.1) *for $\gamma = \rho = 0$. Suppose the function f_1 satisfies the following property: for each $x > 0, a > 1$ there exists t_0 such that for $t \geq t_0$*

$$f(tx/a) - f(t/a) < f_1(tx) - f_1(t) < f(tax) - f(ta),$$

then

$$f_1 - f = o(c) \qquad (t \to \infty),$$

so that f_1 satisfies (2.1) *with the same functions a and c as the function f and with $\gamma = \rho = 0$.*

Remark. Relation (2.30) means that for all $x > 0$ the functions

$$g_x^{(1)}(t) := f_1(tx) - f_1(t)$$

and $g_x(t) := f(tx) - f(t)$ are inversely asymptotic ($g_x^{(1)} \overset{*}{\sim} g_x$; see Geluk and de Haan [6, p. 32]).

Proof.

$$\frac{f_1(tx) - f_1(t) - a(t)\log x}{c(t)}$$
$$< \frac{f(tax) - f(t) - a(t)\log(ax)}{c(t)} - \frac{f(ta) - f(t) - a(t)\log a}{c(t)}$$
$$\to \frac{(\log ax)^2 - (\log a)^2}{2}$$

($t \to \infty$) and the right-hand side tends to $(\log x)^2/2$ as $a \downarrow 1$. A similar lower inequality is easily obtained.

Remark. Similar statements can be made for $\gamma \neq 0$ and/or $\rho \neq 0$, based on Theorem 2 [5]. They differ from case to case and are rather complicated so they are omitted.

Inverses

Property 5. Suppose f is nondecreasing, $\lim_{t\to\infty} f(t) =: f(\infty) \leq \infty$ and g is its right-continuous or left-continuous inverse function. Then (2.11) is equivalent to

$$\lim_{t \uparrow f(\infty)} \left(\left(\frac{g(t + x\alpha(t))}{g(t)} - (1 + \gamma x)^{1/\gamma} \right) \Big/ \beta(t) \right)$$
$$= -(1 + \gamma x)^{(1/\gamma)-1} H\left((1 + \gamma x)^{1/\gamma}\right)$$
$$= -(1 + \gamma x)^{(1/\gamma)-1} \int_0^x \int_0^s (1 + \gamma u)^{\rho/\gamma - 1}\, du\, ds$$
$$= -(1 + \gamma x)^{1/\gamma} \int_0^x (1 + \gamma s)^{-2} \int_0^s (1 + \gamma u)^{\rho/\gamma}\, du\, ds, \quad (2.31)$$

locally uniformly for $x \in (-1/\max(0, \gamma), -1/\max(0, -\gamma))$ with $\alpha(t) := a(g(t))$ and $\beta(t) = c(g(t))$.

Proof. This is Theorem 3 of de Haan and Stadtmüller [5]. The last equality can be checked by applying the operator

$$\frac{d}{dx}(1+\gamma x)^2 \frac{d}{dx}$$

to both sides.

THEOREM 5. *Suppose the function g is twice differentiable and let $g' > 0$. Set*

$$g_1 := \frac{g}{g'}.$$

Suppose further that

$$\lim_{t \uparrow f(\infty)} g_1'(t) = \gamma \tag{2.32}$$

and

$$\lim_{t \uparrow f(\infty)} \frac{\gamma - g_1'(t + x g_1(t))}{\gamma - g_1'(t)} = (1+\gamma x)^{\rho/\gamma} \tag{2.33}$$

locally uniformly on $(-1/\max(0,\gamma), 1/\max(0,-\gamma))$ *for some* $\gamma \in \mathbb{R}$, $\rho \le 0$. *Then*

$$\lim_{t \uparrow f(\infty)} \frac{g(t + x g_1(t))/g(t) - (1+\gamma x)^{1/\gamma}}{\gamma - g_1'(t)}$$

$$= (1+\gamma x)^{1/\gamma} \int_0^x (1+\gamma s)^{-2} \int_0^s (1+\gamma u)^{\rho/\gamma}\, du\, ds. \tag{2.34}$$

Conversely, suppose (2.31) *holds for some* α, β, γ, *and* ρ; *lim*$_{t \uparrow f(\infty)}$ $\beta(t) = 0$; *and the function* g_1 *is monotone, then* (2.17) *and* (2.18) *hold.*

Proof. In de Haan and Resnick [4, Prop. 2.2] it is proved that the conditions on g_1 are fulfilled if and only if the conditions on A_1 in Theorem 1 are fulfilled. What remains is to show the specifics of (2.34). So suppose the conditions on g_2 are fulfilled. Now $\lim_{t \uparrow f(\infty)} g_1'(t) = \gamma$ implies

$$\lim_{t \uparrow f(\infty)} \frac{g(t + x g_1(t))}{g(t)} = (1+\gamma x)^{1/\gamma} \tag{2.35}$$

locally uniformly. Hence

$$\frac{g(t + x g_1(t))}{g(t)} - (1+\gamma x)^{1/\gamma} \sim (1+\gamma x)^{1/\gamma} \{\log g(t + x h_1(t))$$

$$- \log g(t) - \gamma^{-1} \log(1+\gamma x)\}$$

$(t \uparrow f(\infty))$. Write $S := \log g$. Then $g_1(t) = 1/S'(t)$ and

$$(1+\gamma x)^{-1/\gamma} \frac{g(t + x g_1(t))/g(t) - (1+\gamma x)^{1/\gamma}}{\gamma - g_1'(t)}$$

$$\sim \frac{S(t + x g_1(t)) - S(t) - \gamma^{-1}\log(1+\gamma x)}{\gamma - g_1'(t)}$$

$$= \int_0^x \left(\frac{S'(t + s g_1(t))}{S'(t)} - \frac{1}{1+\gamma s} \right) ds / (\gamma - g_1'(t))$$

$$= \int_0^x \frac{S'(t + s g_1(t))}{S'(t)(1+\gamma s)} \int_0^s \frac{\gamma - g_1'(t + u g_1(t))}{\gamma - g_1'(t)} \, du \, ds$$

$$\to \int_0^x (1+\gamma s)^{-2} \int_0^s (1+\gamma u)^{\rho/\gamma} \, du \, ds.$$

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