A simple approach to discrete-time infinite horizon problems

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Abstract. In this note, we consider a type of discrete-time infinite horizon problem that has one ingredient only, a constraint correspondence. The value function of a policy has an intuitive monotonicity property; this is the essence of the four standard theorems on the functional equation ('the Bellman equation'). Some insight is offered into the boundedness condition for the value function that occurs in the formulation of these results: it can be interpreted as accountability of the loss of value caused by a non-optimal policy or, alternatively, it can be interpreted as irrelevance of deviations, in the distant future, from the considered policy. Without the boundedness condition, there is a gap, which can be viewed as the persistent potential positive impact of deviations, in the distant future, from the considered policy. The general stationary discrete-time infinite horizon optimization problem considered in Stokey and Lucas (1989) can be mapped to this type of problems and so the results in the present paper can be applied to this general class of problems.

Introduction

The most complete treatment of stationary discrete-time infinite horizon maximization problems is in Stokey and Lucas (1989). In the present note, a different discrete-time infinite horizon problem will be studied. We map the former problem to the latter. This shows that the former problem is essentially a special case of the latter. The latter contains non-stationary problems as well. We formulate for the latter problem the four standard results: the necessary and sufficient conditions in terms of the functional equation ('the Bellman equation'), for the value function as well as for optimal policies. These imply immediately the corresponding results for the former problem that are formulated and proved in Stokey and Lucus (1989).

The latter problem is of interest by virtue of the simplicity of its formulation and analysis. It is determined by one ingredient only, the constraint correspondence. Moreover, if we carry out an admissible policy and consider at each moment the optimal value for the remaining problem, we get a decreasing ('monotonically non-increasing') function of time, and optimal policies are characterized by the property that this function is constant. These properties immediately suggest the statements of the four standard results as well as their proofs. These proofs fall as it were as ripe apples from the tree; one does not have to make an effort to construct these. We give insight into the boundedness condition that occurs in the formulation of these results: it represents the reasonable assumption that the loss of value caused by a non-optimal policy is the sum of the losses of value caused by the individual decisions of the policy. Alternatively, it represents the reasonable assumption that decisions taken in the distant future are irrelevant. Without the boundedness condition, there is a gap, which represents the persistent potential positive impact of deviations in the distant future from the policy under consideration.

We restrict attention in this paper to deterministic problems for the sake of simplicity, but everything extends to stochastic problems: then the monotonicity property has to be replaced by a supermartingale property and the constancy property by a martingale property.

1 Two types of problems and their relation

The general discrete-time infinite horizon optimization problem. We recall the general discrete-time infinite horizon maximization problem (see chapter 4 from Stokey and Lucas (1989)), also called the sequential problem. To be precise, it is the following family of problems $(SP_{x_0})_{x_0 \in X}$ (or, equivalently, the problem (SP_{x_0}) with parameter x_0 running over X):

$$(SP_{x_0}) \qquad \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

$$\text{s.t.} \quad x_{t+1} \in \Gamma(x_t), \quad t = 0, 1, 2, \ldots,$$

$$x_0 \in X \text{ given.}$$

Here $X \subseteq \mathbb{R}^n$, $\beta \in (0,1), F : X \times X \to \mathbb{R}, \Gamma : X \rightrightarrows X$ (a correspondence, that is, Γ associates to each element of X a subset of X). It is assumed that $\Gamma(x)$ is nonempty for all $x \in X$, and that for each $x_0 \in X$ and each admissible element $\{x_{t+1}\}_{t=0}^{\infty}$ of problem (SP_{x_0}), the series $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$ converges within $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ (that is, we allow the sum of the series to be $+\infty$ or $-\infty$).

Problem associated to a constraint correspondence. In the present note, we consider a different problem. Let for each $t \in \mathbb{N}_0 = \{0, 1, \ldots\}$ a correspondence $\Gamma_t : X \rightrightarrows X$ be given. We assume that $\Gamma_t(x)$ is nonempty for all $(t, x) \in \mathbb{N}_0 \times X$. For each $(t_0, x_0) \in \mathbb{N}_0 \times X$, we write $\Pi(t_0, x_0)$ for the set of all infinite sequences $\mathbf{x} = \{x_{t+1}\}_{t=t_0}^{\infty}$ in X for which $x_{t+1} \in \Gamma_t(x_t)$ for all $t \in \mathbb{N}_0$ with $t \geq t_0$, where $x_{t_0} = x_0$. We assume that for each $(t_0, x_0) \in \mathbb{N}_0 \times X$ and each sequence $\mathbf{x} \in \Pi(t_0, x_0)$, the sequence of last coordinates, $\{(x_{t+1})_n\}_{t=t_0}^{\infty}$, converges with limit $l(\mathbf{x}) = \lim_{t\to\infty}(x_{t+1})_n$ in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ (that is, we allow the limit to be $+\infty$ or $-\infty$). We consider the family of maximization problems $(P_{t_0,x_0})_{(t_0,x_0)\in\mathbb{N}_0\times X}$ defined by

$$l(\mathbf{x}) \to \max, \mathbf{x} \in \Pi(t_0, x_0). \tag{P_{t_0,x_0}}$$

That is, (P_{t_0,x_0}) is the problem to choose recursively $x_{t_0} = x_0, x_{t+1} \in \Gamma_t(x_t)$ for $t = t_0, t_0+1, \ldots$ such that the last coordinate of the vectors x_{t+1} tends to a maximal limit for $t \to \infty$. We note that this family of problems is determined by one ingredient only, the constraint correspondence $\Gamma : \mathbb{N}_0 \times X \rightrightarrows X : (t, x) \mapsto \Gamma_t(x)$.

The relation between the two types. Now we map the former problem into the latter, that is, we reformulate the former problem as a latter problem. The former problem is given by the ingredients $X \subseteq \mathbb{R}^n$, $\beta \in (0, 1), F : X \times X \to \mathbb{R}, \Gamma : X \rightrightarrows X$. We map this to the latter problem that is given by the following constraint correspondence $\tilde{\Gamma}$. Let $\tilde{X} = X \times \mathbb{R}$ and let, for each $t \in \mathbb{N}_0$, the correspondence $\tilde{\Gamma}_t : \tilde{X} \rightrightarrows \tilde{X}$ be given by $\tilde{\Gamma}_t(x, y) = \{(\bar{x}, \bar{y}) | \bar{x} \in \Gamma(x), \bar{y} = y + \beta^t F(x, \bar{x})\}$ for all $x \in X$ and $y \in \mathbb{R}$. One has, for each $t_0 \in \mathbb{N}_0$ and each $x_0 \in X$, a bijection from the admissible set of (SP_{x_0}) to the admissible set of $(P_{t_0,\tilde{x}_0})$, where $\tilde{x}_0 = (x_0, y_0) \in \tilde{X}$ with $y_0 = 0$, by $\{x_{t+1}\}_{t=0}^{\infty} \mapsto \{\tilde{x}_{t+1}\}_{t=t_0}^{\infty} = \{(x_{t+1}, y_{t+1})\}_{t=0}^{\infty}$, where $y_{t+1} = y_t + \beta^{t+t_0} F(x_t, x_{t+1}), \quad t = 0, 1, \ldots$. The value of the objective function of the latter element $\{\tilde{x}_{t+1}\}_{t=t_0}^{\infty}$ in the latter problem $(P_{t_0,\tilde{x}_0})$ equals β^{t_0} times the value of the objective function of the former element $\{x_{t+1}\}_{t=0}^{\infty}$ in the former problem (SP_{x_0}). Indeed, one has $l(\{\tilde{x}_{t+1}\}_{t=t_0}^{\infty}) = \lim_{t\to\infty} y_{t+1} = \lim_{t\to\infty} \sum_{\tau=0}^{t} \beta^{t_0+\tau} F(x_\tau, x_{\tau+1}) = \beta^{t_0} \sum_{\tau=0}^{\infty} \beta^{\tau} F(x_\tau, x_{\tau+1})$

Therefore, these two problems are equivalent.

2 The standard results for the problem associated to a constraint correspondence

The first standard result: the functional equation. We consider the family of problems $(P_{t_0,x_0})_{(t_0,x_0)\in\mathbb{N}_0\times X}$ that has been associated, in the previous section, to a constraint correspondence $\Gamma : \mathbb{N}_0 \times X \rightrightarrows X$. Let $v^* : \mathbb{N}_0 \times X \to \overline{\mathbb{R}}$ be the value function of this family—$v^*(t_0, x_0) = \sup_{\mathbf{x} \in \Pi(t_0, x_0)} l(\mathbf{x})$ for all $(t_0, x_0) \in \mathbb{N}_0 \times X$. We consider the following functional equation for functions $v : \mathbb{N}_0 \times X \to \overline{\mathbb{R}}$:

$$v(t, x) = \sup_{y \in \Gamma_t(x)} v(t+1, y). \qquad (FE)$$

Theorem 1. *The value function v^* of the family of problems $(P_{t_0,x_0})_{(t_0,x_0) \in \mathbb{N}_0 \times X}$ associated to a constraint correspondence Γ satisfies the functional equation (FE).*

Proof. To prove the theorem, it suffices to show that for each $(t_0, x_0) \in \mathbb{N}_0 \times X$ one has the equality $v^*(t_0, x_0) = \sup_{y \in \Gamma_{t_0}(x_0)} v^*(t_0 + 1, y)$. The left hand side of this equality is the supremum of $l(\mathbf{x})$ over all $\mathbf{x} \in \Pi(t_0, x_0)$. Now we calculate, to begin with, the outcome of taking the supremum of $l(\mathbf{x})$ only over all sequences $\mathbf{x} \in \Pi(t_0, x_0)$ for which the second term, x_{t_0+1}, equals an arbitrary fixed element $y \in \Gamma_{t_0}(x_0)$. If we delete from these sequences the first element, we get precisely the sequences of $\Pi(t_0 + 1, y)$; moreover, deleting the first element of the convergent sequence $\mathbf{x}$ does not alter its limit $l(\mathbf{x})$, of course. Therefore, this calculation has outcome $v^*(t_0 + 1, y)$. It remains to take the supremum of this over all $y \in \Gamma_{t_0}(x_0)$ and to observe that this is precisely the righthand side of the required equality.

Corollary. *For each $(t_0, x_0) \in \mathbb{N}_0 \times X$ and each admissible element $\boldsymbol{x} = \{x_{t+1}\}_{t \geq t_0}$ of problem $P_{(t_0,x_0)}$, the sequence $\{v^*(t, x_t)\}_{t=t_0}^{\infty}$ is decreasing ('monotonically non-increasing'). Moreover, $\boldsymbol{x}$ is optimal if and only if this sequence is constant.*

It is this monotonicity that is responsible for the simplicity of the analysis of our problem.

Accountability of loss of value. In the remaining three standard theorems, the boundedness condition

$$\lim_{t \to \infty} (v(t, x_t) - (x_t)_n) = 0$$

for all $(t_0, x_0) \in \mathbb{N}_0 \times X$ and all $\mathbf{x} \in \Pi(t_0, x_0)$ plays a role. This equality implies the equality $\lim_{t \to \infty} v^*(t, x_t) = l(\mathbf{x})$. If $l(\mathbf{x})$ is finite, the two equalities are equivalent. Now we give an interpretation of this condition that is more intuitive: the accountability of loss of value. We assume to begin with, for simplicity of explanation, that all problems (P_{t_0,x_0}) are solvable. An element $\mathbf{x} \in \Pi(0, x_0)$ will be called a policy (to be more precise: an admissible policy). A choice of policy $\mathbf{x}$ 'causes' a loss of value $v^*(0, x_0) - l(\mathbf{x})$, compared to the optimal value. Indeed, by choosing the policy $\mathbf{x}$, one forsakes the opportunity to reach value $v^*(0, x_0)$, and achieves instead only value $l(\mathbf{x})$. Thus choosing policy $\mathbf{x}$ leads to a loss of value $v^*(0, x_0) - l(\mathbf{x})$. The part of this loss that can be attributed to the t-th decision of this policy, x_t, is the difference

$v^*(t-1, x_{t-1}) - v^*(t, x_t)$. Indeed, before taking decision x_t, one is confronted with problem $(P_{t-1,x_{t-1}})$, which has value $v^*(t-1, x_{t-1})$ and after taking decision x_t, one is confronted with problem (P_{t,x_t}), which has value $v^*(t, x_t)$. Thus, by taking this decision, one forsakes the opportunity to achieve value $v^*(t-1, x_{t-1})$ and one is in a position that the best value that can be reached is $v^*(t, x_t)$. Hence, taking decision x_t leads to a loss of value $v^*(t-1, x_{t-1}) - v^*(t, x_t)$. Summing up all losses of value caused by the individual decisions that make up the policy $\mathbf{x}$ one gets $v^*(0, x_0) - \lim_{t\to\infty} v^*(t, x_t)$. It is reasonable to assume that the entire loss of value caused by choosing policy $\mathbf{x}$, that is, $v^*(0, x_0) - l(\mathbf{x})$, can be accounted for by the losses caused by the individual decisions x_t that make up the policy $\mathbf{x}$. This assumption is now seen to be the equality $\lim_{t\to\infty} v^*(t, x_t) = l(\mathbf{x})$, which is precisely the boundedness condition defined above. Thus the boundedness condition can be viewed as the assumption that *the loss of value caused by choosing policy* $\mathbf{x}$ *can entirely be accounted for by the losses of value caused by the individual decisions that make up the policy.* If we do not assume the boundedness condition, then part of the loss of value might not be accounted for by the losses caused by individual decisions. This part or gap can be interpreted as *the persistent potential positive impact of deviation, in the distant future, from the considered policy.* Indeed, if this gap equals $G > 0$, then for each $\varepsilon > 0$ and for every moment T however far in the future (therefore the word persistent has been used), there exists a policy that deviates from the given policy only beyond moment T and that achieves a value that is at least $G - \varepsilon$ higher than the value that is achieved by the considered policy.

Now we consider the more general situation that the problems (P_{t_0,x_0}) is not necessarily solvable for all pairs (t_0, x_0). Take $(t_0, x_0) \in \mathbb{N}_0 \times X$ and an admissible element $\mathbf{x}$ for problem (P_{t_0,x_0}). The 'loss of value' of taking $\mathbf{x}$ compared to the optimal value $v^*(t_0, x_0)$ of the problem, is $v^*(t_0, x_0) - l(\mathbf{x}) \in [0, \infty]$. We assume that this is well-defined, that is, we exclude that either $v^*(t_0, x_0) = l(\mathbf{x}) = \infty$ or $v^*(t_0, x_0) = l(\mathbf{x}) = -\infty$. Now we address the natural question how to account for this loss, that is, how to attribute it to clear causes. The limit $\lim_{t\to\infty} v^*(t, x_t) \in \overline{\mathbb{R}}$ exists by the corollary to theorem 1, as every decreasing sequence in $\overline{\mathbb{R}}$ converges. Each jump down in the sequence $\{v^*(t, x_t)\}_{t=t_0}^{\infty}$, that is, $v^*(\bar{t}, x_{\bar{t}}) > v^*(\bar{t}+1, x_{\bar{t}+1})$, means that the decision at time $\bar{t}$ to go from $x_{\bar{t}}$ to $x_{\bar{t}+1}$ is not optimal, given the situation $(\bar{t}, x_{\bar{t}})$. Moreover, all terms of the sequence have lower bound $l(\mathbf{x})$: for each $\bar{t} \in \mathbb{N}_0$, the sequence $\{x_t\}_{t=\bar{t}}^{\infty}$ is in $\Pi(\bar{t}, x_{\bar{t}})$ and its limit is $l(\mathbf{x})$, so $v^*(\bar{t}, x_{\bar{t}}) \geq l(\mathbf{x})$. It follows that $\lim_{t\to\infty} v^*(t, x_t) \geq l(\mathbf{x}) = \lim_{t\to\infty}(x_t)_n$. If this last inequality would be sharp, it would mean that $\mathbf{x}$ leads, apart from the losses of value associated to the jumps, which were mentioned above, to an additional loss of value, the gap $\lim_{t\to\infty} v^*(t, x_t) - l(\mathbf{x}) \in (0, \infty]$, or equivalently, $\lim_{t\to\infty}(v^*(t, x_t) - (x_t)_n) \in (0, \infty)$. This loss is hard to attribute to a clear cause. It cannot be accounted for by the losses of the individual decisions. Therefore, it is tempting to exclude such an unaccountable loss. This line of thought leads to the boundedness condition.

Now we state explicitly the interpretation of the boundedness condition that we have just discussed. The total loss of value caused by the policy $\mathbf{x}$, $v^*(t_0, x_0) - l(\mathbf{x})$, is equal to the sum of the losses of value $v^*(t, x_t) - v^*(t+1, x_{t+1})$ where t runs over all integers $\geq t_0$ for which $v^*(t, x_t) > v^*(t+1, x_{t+1})$. Therefore, for each of these t we can say precisely how much the decision to go from x_t to x_{t+1} 'costs' in terms of lost value: $v^*(t, x_t) - v^*(t+1, x_{t+1})$. Thus the total loss $v^*(t_0, x_0) - l(\mathbf{x})$ is completely accounted for. So we can view the equality $\lim_{t\to\infty} v^*(t, x_t) = l(\mathbf{x})$ as *an assumption on the accountability of loss of value.*

Irrelevance of decisions in the long run. Now we give a second intuitive interpretation of the boundedness condition: the irrelevance of decisions in the long run. The equality $\lim_{t\to\infty}(v^*(t, x_t) - (x_t)_n) = 0$ suggests still another interpretation: for each admissible element $\mathbf{x}$, it makes not much difference for the value of the objective function, $l(x)$, whether decisions in the far future are optimal or not. So we can view the equality $\lim_{t\to\infty}(v^*(t, x_t) - (x_t)_n) = 0$ as *an assumption on the irrelevance of decisions in the far future.*

The second standard result: unique solution for the functional equation

Uniqueness theorem for the functional equation. Now we consider along with the functional equation (FE) the following boundary condition

$$\lim_{t\to\infty}(v(t, x_t) - (x_t)_n) = 0$$

for all $(t_0, x_0) \in \mathbb{N}_0 \times X$ and all $\mathbf{x} \in \Pi(t_0, x_0)$.

Theorem 2. *Under the assumption on irrelevance of decisions in the future/accountability of loss of value, that is,*$\lim_{t\to\infty}(v^*(t, x_t) - (x_t)_n) = 0$*, the value function* v^* *is the unique solution of* (FE) *that satisfies the boundary condition.*

Proof. The value function v^* satisfies (FE) by theorem 1 and it satisfies the boundary condition by the additional assumption on the accountability of loss of values. Conversely, assume that v satisfies the functional equation and the boundary condition. Take $(t_0, x_0) \in \mathbb{N}_0 \times X$. To prove the theorem, it suffices to verify the inequalities $v(t_0, x_0) \geq v^*(t_0, x_0)$ and $v(t_0, x_0) \leq v^*(t_0, x_0)$. For each $\mathbf{x} \in \Pi(t_0, x_0)$, one has, by repeated use of (FE), that $v(t_0, x_0) \geq v(t_0+1, x_{t_0+1}) \geq v(t_0+2, x_{t_0+2}) \geq \cdots$. By the boundary condition, this sequence converges to $l(\mathbf{x})$. As $\mathbf{x} \in \Pi(t_0, x_0)$ is arbitrary, it follows from the definition of $v^*(t_0, x_0)$ that $v(t_0, x_0) \geq v^*(t_0, x_0)$. To prove the converse inequality, we have to distinguish three cases. Firstly, if $v(t_0, x_0) = -\infty$, then it follows from $v(t_0, x_0) \geq v^*(t_0, x_0)$ that $v^*(t_0, x_0) = -\infty$.

Secondly, consider the case that $v(t_0, x_0)$ is finite. Then, again by repeated use of (FE) there exists for each $\varepsilon > 0$ an element $\mathbf{x} \in \Pi(t_0, x_0)$ for which $v(t_0, x_0) \leq v(t_0 + 1, x_{t_0+1}) + \frac{1}{2}\varepsilon \leq (v(t_0 + 2, x_{t_0+2}) + \frac{1}{4}\varepsilon) + \frac{1}{2}\varepsilon \leq \cdots$. By the boundary condition, this sequence converges to $l(\mathbf{x}) + \varepsilon$. This implies, by $l(\mathbf{x}) \leq v^*(t_0, x_0)$, that $v(t_0, x_0) \leq v^*(t_0, x_0) + \varepsilon$. As $\varepsilon > 0$ is arbitrary, it follows that $v(t_0, x_0) \leq v^*(t_0, x_0)$. Thirdly, consider the case that $v(t_0, x_0) = \infty$. Choose $k \in \mathbb{N}_0$ and choose recursively $x_i \in X, 1 \leq i \leq k$ such that $x_i \in \Gamma(x_{i-1})$ for $1 \leq i \leq k$ and $v(t_i, x_i) = \infty$ for $1 \leq i \leq k$ and $v(k+1, x_{k+1})$ is finite for each $x_{k+1} \in \Gamma_k(x_k)$. This is possible, otherwise one would get a contradiction with the boundary condition. Indeed, the equality $\lim_{t\to\infty}(v(t, x_t) - (x_t)_n) = 0$ implies that there exists $\bar{t} \geq t_0$ such that $v(t, x_t)$ is finite for all $t \geq \bar{t}$. There exists, by $v(k, x_k) = \infty$ and (FE), for each $N > 0$ an element $x_{k+1} \in \Gamma_k(x_k)$ such that $v(k+1, x_{k+1}) \geq N$. As $v(k+1, x_{k+1})$ is finite, we have $v(k+1, x_{k+1}) = v^*(k+1, x_{k+1})$ by what we have proved above. So $v^*(k + 1, x_{k+1}) \geq N$ and so, by the corollary, $v^*(t_0, x_0) \geq N$. As $N > 0$ is arbitrary, $v^*(t_0, x_0) = \infty$, as required.

Third and fourth standard result: characterization of optimal policies

Now we give a characterization of optimal solutions; this corresponds to the third and fourth standard result.

Proposition. *Under the assumption on irrelevance of decisions in the future/accountability of loss of values, an admissible element of problem (P_{t_0,x_0}) is optimal if and only if the sequence $\{v^*(t, x_t)\}_{t=0}^{\infty}$ is constant.*

Proof. Assume that the sequence $\{v^*(t, x_t)\}_{t=0}^{\infty}$ is constant. Then $v^*(t_0, x_0) = (v^*(t, x_t) - (x_t)_n) + (x_t)_n$ for all $t \in \mathbb{N}_0$. Taking the limit $t \to \infty$ and using the assumption on accountability of loss of values, we get $v^*(t_0, x_0) = l(\mathbf{x})$, that is, $\mathbf{x}$ is optimal.

Remark. One can replace the assumption in the proposition by the weaker one that

$$\limsup_{t\to\infty}(v^*(t, x_t) - (x_t)_n) = 0$$

for each $(t_0, x_0) \in \mathbb{N}_0 \times X$ and each $\mathbf{x} \in \Pi(t_0, x_0)$. The proof remains the same up to a small adjustment: now one gets $v^*(t_0, x_0) \leq l(\mathbf{x})$. As the admissible $\mathbf{x}$ is admissible, the inverse inequality holds as well, and so one has $v^*(t_0, x_0) = l(\mathbf{x})$.

3 Comparison to the results and proofs in Stokey and Lucas (1989)

Consider the general discrete-time infinite horizon maximization problem or sequential problem, that is, the family of problems $(SP_{x_0})_{x \in X}$, defined in section 1. In section 1 we have seen that this can be mapped to an equivalent problem that is associated to a constraint correspondence. Therefore, one can apply the results in section 2, and then one can translate back in terms of the original formulation of the sequential problem. Let $v^*(x_0)$ be the value of (SP_{x_0}). Theorem 1 gives that the value function v^* satisfies the functional equation in Stokey and Lucas (1989), $v(x) = \sup_{y \in \Gamma(x)}[F(x,y) + \beta v(y)]$, all $x \in X$. The assumption in section 2 that $l(\mathbf{x})$ exists gives here: $\lim_{n \to \infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists; this is assumption 4.2 in Stokey and Lucas (1989); the other assumption 4.2, $\Gamma(x) \neq \emptyset$ also corresponds. The results in section 4.1 of Stokey and Lucas (1989) (pp. 66-77) follow from the theorems 1 and 2. Applying theorem 1 respectively theorem 2 to the sequential problem gives theorem 4.2 respectively theorem 4.3. Applying the second statement of the corollary gives theorem 4.4, and applying the proposition and the remark following it gives theorem 4.5. The proofs in section 2 are based on the same principles as the proofs in Stokey and Lucas (1989). However, they have some advantages over those in Stokey and Lucas (1989): the problem that we consider requires only one ingredient, the constraint correspondence Γ, and it has a monotonicity property, and as a consequence the proofs flow naturally: these fall as it were ‘like ripe apples from the tree’. Moreover, the analysis of the problem that we consider leads to some novel insight into the boundary condition of the functional equation (FE): interpretations, which show that this condition is a natural one to assume. Moreover, the results in section 2 are more general than in Stokey and Lucas (1989), they hold for non-stationary problems as well. The analysis in this paper extends to stochastic problems: then ‘decreasing functions’ respectively ‘constant functions’ in theorem 1 have to be replaced by supermartingales respectively martingales.

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Reference

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