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A note on “A multi-period profit maximizing model for retail supply chain management”

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ABSTRACT

In this note we present an efficient exact algorithm to solve the joint pricing and inventory problem for which Bhattacharjee and Ramesh (2000) proposed two heuristics. The algorithm is based on a method proposed by Thomas (1970) and we show additional properties which can be used to arrive at an even more efficient algorithm. Furthermore, we point out several shortcomings in the paper by Bhattacharjee and Ramesh.

1. Introduction

Bhattacharjee and Ramesh (2000) consider a joint pricing and inventory model for a monopolistic retailer who is dealing in a single product. In the pricing and inventory problem, the retailer wants to maximize his profit considering revenue and all relevant costs for a given planning horizon. Bhattacharjee and Ramesh propose two heuristic algorithms and an exact approach that runs in exponential time to solve this problem. According to Google Scholar the paper has received more than 100 citations, including recent ones, implying that the topic is still of interest. Instead of solving the problem either inefficiently or heuristically, we show in this note that the problem can be solved to optimality in polynomial time, that is, in an efficient way. We do this by applying a method already proposed by Thomas (1970) for a similar problem. Furthermore, we prove additional properties of optimal solutions, which can be used to arrive at an even more efficient algorithm. Finally, we point out shortcomings in the modeling and analysis in the paper of Bhattacharjee and Ramesh.

Although there are similar models in the literature, to the best of our knowledge, the lot-sizing and pricing model under consideration is not a special case of any other existing model, implying that our results cannot be directly obtained from the existing literature. To position our work, we briefly describe some related works from the literature. Deng and Yano (2006) consider a joint lot-sizing and pricing model with production capacities. Although the model of Deng and Yano (2006) is more general in terms of capacities, it does not consider lower and upper bounds on the prices. Furthermore, Geunes, Romeijn, and Taaffe (2006) consider a model with a set of customers in each period. Each customer demand can be partly served and the revenue of a customer depends linearly on the amount served, which leads to a piecewise linear concave revenue function. Again, this is different from the model under consideration, where the revenue function does not have this particular structure. We note that the approach of Geunes et al. (2006) could be used to solve the problem under consideration by approximating the revenue function by a piecewise linear function, but this would result in a loss of precision and efficiency.

The remainder of this note is organized as follows. In Section 2 we describe the joint pricing and inventory model and we give a mathematical formulation. In Section 3 we present the exact method proposed by Thomas (1970), apply this method to the Bhattacharjee and Ramesh case, and show how the running time can be further improved. In Section 4 we point out the shortcomings in the main results presented by Bhattacharjee and Ramesh.

2. Problem description

Bhattacharjee and Ramesh consider the following joint pricing and inventory model. There is a monopolistic retailer dealing in a single product over a finite time horizon. At the beginning of each period ordering and pricing decisions are made. This means that in each period a different price can be set. For each order made by the retailer there is a fixed ordering cost and variable purchasing cost. Holding cost is incurred for carrying inventory from a period to the next period.

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Furthermore, it is assumed in the paper that demand satisfies the following equation

$$d(p) = \beta p^{-\alpha}, \tag{1}$$

where β is a constant, p is the price and $\alpha > 1$ is the demand elasticity. Finally, it is assumed that price in each period t satisfies $p_{\min} \le p_t \le p_{\max}$. Assuming that all demand has to be satisfied (i.e., loss of demand is not allowed) and using the following notation,

T = model horizon
K = fixed ordering cost
c = per unit purchase cost
h = holding costs per unit per period
p_t = price set in period t
q_t = ordered quantity in period t
I_t = ending inventory in period t,

the problem can be formulated as follows

$$\begin{array}{lll} \max & \sum_{t=1}^{T} d(p_t)p_t - C(D(p)) & \\ \text{s.t.} & p_{\min} \le p_t \le p_{\max} & t = 1, \ldots, T \end{array} \tag{2}$$

where

$$\begin{array}{llll} C(D(p)) = & \min & \sum_{t=1}^{T} (K\delta(q_t) + cq_t + hI_t) & \\ & \text{s.t.} & I_t = I_{t-1} - d(p_t) + q_t & t = 1, \ldots, T \\ & & q_t, I_t \ge 0 & t = 1, \ldots, T \\ & & I_0 = 0 & \end{array}$$

with

$$\delta(x) = \begin{cases} 0 & for\ x = 0 \\ 1 & for\ x > 0 \end{cases}$$

and $D(p)$ is the demand vector $D(p) = [d(p_1), \ldots, d(p_T)]$.

In problem (2) we maximize the total revenue minus total cost over all periods, such that price is bounded from above and below. If we set $p_{\min} = 0$ and $p_{\max} = \infty$, then price is not restricted in the model. The total cost is represented by $C(D(p))$, which is a 'standard' Wagner–Whitin problem (see Wagner & Whitin, 1958). We minimize ordering, purchasing and holding cost, such that demand is satisfied and order quantity and ending inventory are nonnegative in each period. Furthermore, we may assume without loss of generality that starting inventory is zero.

3. Solution approach

3.1. General solution approach

In this section we propose an exact algorithm that has a running time which is quadratic in the model horizon T. This method was proposed by Thomas (1970) for a similar problem. Thomas considers a more general problem, where the demand functions and the cost parameters may vary over time. The proposed method (in the general case) is explained below. Note that Thomas presented the model as a cost minimization problem, whereas we present it as a profit maximization model.

For $1 \le j \le t \le T$ define p_{jt} as the price vector $p_{jt} = [p_j, \ldots, p_t]$ and define $\pi_{jt}(p_{jt})$ as the total profit if production takes place in period j to satisfy demands in periods $j, \ldots, t$ (we will call this a subplan), i.e.,

$$\pi_{jt}(p_{jt}) = \sum_{k=j}^{t} \left(p_k - c_j - \sum_{i=j}^{k-1} h_i \right) d_k(p_k) - K_j. \tag{3}$$

Furthermore, define π_{jt} as the maximum profit for a subplan consisting of periods $j, \ldots, t$, i.e.,

$$\pi^*_{jt} = \max_{p_{jt}} \pi_{jt}(p_{jt}). \tag{4}$$

Thomas shows that if a setup takes place in period j and the next setup in period t, then the optimal price for period $k = j, \ldots, t-1$ must be set at the value which maximizes

$$\left(p_k - c_j - \sum_{i=j}^{k-1} h_i \right) d_k(p_k).$$

Dependent on the structure of $d_t(p_t)$ we can calculate this optimal price in an analytical way or, if necessary, by a numerical procedure. Substituting the optimal prices in (3) we are able to determine π^*_{jt}. Because it can be shown that the optimal solution consists of a series of consecutive subplans, the following forward recursion enables us to find the optimal profit for the whole model horizon:

$$F(t) = \max_{j=1,\ldots,t} (F(j-1) + \pi^*_{jt}) \text{ for } t = 1, \ldots, T \text{ with } F(0) = 0. \tag{5}$$

3.2. The Bhattacharjee and Ramesh case

For the Bhattacharjee and Ramesh case we can find the optimum of (3) in an analytical way. Substituting demand function (1) and the constant cost parameters (i.e., $K_t = K$, $c_t = c$ and $h_t = h$ for $t = 1, \ldots, T$) in (3) we have that

$$\begin{aligned} \pi_{jt}(p_{jt}) &= \sum_{k=j}^{t} \left[p_k - c - \sum_{i=j}^{k-1} h \right] \beta {p_k}^{-\alpha} - K \\ &= \sum_{k=j}^{t} [p_k - c - (k-j)h] \beta {p_k}^{-\alpha} - K. \end{aligned} \tag{6}$$

Calculating the first order conditions we have for the subplan consisting of periods $i = j, \ldots, t$

$$\frac{\partial \pi_{jt}(p_{jt})}{\partial p_i} = 0 \Leftrightarrow \alpha\beta c {p_i}^{-\alpha-1} + (i-j)h\alpha\beta {p_i}^{-\alpha-1} - (\alpha-1)\beta {p_i}^{-\alpha} = 0$$

or

$$p_i^* = \frac{\alpha(c + (i-j)h)}{\alpha - 1} > 0 \text{ as } \alpha > 1,\ i \ge j \text{ and } c, h \ge 0. \tag{7}$$

Note that p_i^* is not dependent on the other prices set in the periods of the subplan. Furthermore, note that p_i^* does only depend on period j and not on period t, which implies that the optimal price for a single period is only dependent on the starting period of the subplan and independent of the length of the subplan. Finally, one can verify that

$$\left.\frac{\partial \pi_{jt}(p_{jt})}{\partial p_i}\right|_{p_i} > 0 \text{ for } p_i < p_i^* \text{ and } \left.\frac{\partial \pi_{jt}(p_{jt})}{\partial p_i}\right|_{p_i} < 0 \text{ for } p_i > p_i^*,$$

which implies that the maximum profit function for a single period in a subplan is unimodal and that it has a unique optimum at price p_i^*.

If we analyze the second order partial derivative we find

$$\frac{\partial^2 \pi_{jt}(p_{jt})}{\partial {p_i}^2} = -\alpha(\alpha+1)\beta(c + (i-j)h){p_i}^{-\alpha-2} + \alpha(\alpha-1)\beta {p_i}^{-\alpha-1},$$

which is equal to zero for

$$\widehat{p}_i = \frac{(\alpha+1)(c + (i-j)h)}{\alpha - 1} > p_i^*.$$

It is not difficult to verify that the second order partial derivative is smaller than zero for $p_i < \widehat{p}_i$ and larger than zero for $p_i > \widehat{p}_i$. This means that the maximum profit function for a single period in a subplan is concave for $p_i < \widehat{p}_i$ and convex for $p_i > \widehat{p}_i$.

Because Bhattacharjee and Ramesh assume a time-invariant demand function and constant cost parameters, it follows from (3), (4) and (7) that

$$\pi^*_{jt} = \pi^*_{1,t-j+1} \text{ for all } 1 \le j \le t \le T. \quad (8)$$

This means that it is only necessary to evaluate π^*_{1t} for $t = 1, \ldots, T$. Recursive equations (9) and (10) can be used to calculate π^*_{1t} for $t = 1, \ldots, T$ in $\mathcal{O}(T)$ time.

$$p^*_{t+1} = p^*_t + \frac{\alpha h}{\alpha - 1} \quad (9)$$

$$\pi^*_{1,t+1} = \pi^*_{1,t} + (p^*_{t+1} - c - th)\beta {p^*_{t+1}}^{-\alpha} \quad (10)$$

with

$$p^*_1 = \frac{\alpha c}{\alpha - 1} \text{ and } \pi^*_{11} = (p^*_1 - c)\beta {p^*_1}^{-\alpha} - K.$$

By applying recursion formula (5) and using (8), we can find the optimal total profit and the optimal subplans. The optimal prices can be found by using formula (7).

Note that it follows from (9) that the difference between consecutive prices is constant, i.e., $p^*_i - p^*_{i+1} = \frac{\alpha h}{\alpha - 1}$. This property does not only hold for production functions with a setup and unit production cost, but for any production function, as formalized in the next theorem.

Theorem 1. *Consider a subplan* $1, \ldots, t$ *with total demand D. Then, given any production function f, the prices that maximize the profit of the subplan satisfy* $p^*_{i+1} - p^*_i = \frac{\alpha h}{\alpha - 1}$.

Proof. Let f be an arbitrary production function and consider the problem

$$\max \sum_{k=1}^{t} [p_k - (k-1)h]\beta {p_k}^{-\alpha} - f(D)$$
$$\text{s.t. } \sum_{k=1}^{t} \beta {p_k}^{-\alpha} = D,$$

which maximizes the total profit of the subplan. Introducing the Lagrangian multiplier λ, the following set of equations solves the optimization problem:

$$\frac{\partial}{\partial p_i}\left[\sum_{k=1}^{t}(p_k - (k-1)h)\beta {p_k}^{-\alpha} - f(D)\right] + \lambda \frac{\partial}{\partial p_i}\left[D - \sum_{k=1}^{t}\beta {p_k}^{-\alpha}\right] = 0 \quad \text{for } i = 1, \ldots, t,$$
$$\sum_{k=1}^{t} \beta {p_k}^{-\alpha} = D.$$

After some algebra the first set of constraints reduces to

$$p_i = \frac{\alpha h(i-1)}{\alpha - 1} + \lambda \frac{\alpha}{\alpha - 1} \text{ for } i = 1, \ldots, t$$

and it follows that $p^*_{i+1} - p^*_i = \frac{\alpha h}{\alpha - 1}$. □

The proof immediately suggests a way to find the optimal prices corresponding to some aggregate demand level D. Start with some initial value p_1. Because the other prices can be expressed in p_1, i.e. $p_k = \frac{\alpha(k-1)h}{\alpha - 1} + p_1$, solving $\sum_{k=1}^{t} \beta {p_k}^{-\alpha} = D$ reduces to solving

$$\sum_{k=1}^{t} \beta \left(\frac{\alpha(k-1)h}{\alpha - 1} + p_1\right)^{-\alpha} = D,$$

which is an equation in the single variable p_1. Furthermore, because the left hand side is decreasing in p_1, the price p_1 that satisfies the equation can be found by binary search.

As Theorem 1 holds for any aggregate demand level D, it also holds for the optimal demand level. Furthermore, Theorem 1 holds for any production function and subplan. However, this does not imply that the price property also holds for optimal solutions for the Bhattacharjee and Ramesh case with arbitrary production functions. It is well known that the optimal solution consists of consecutive subplans if the production function is concave. Because for general production functions this may not be the case, consecutive prices will not have the property of Theorem 1 in general since the solution may not consist of consecutive subplans.

Because from (9) and (10) it follows that π^*_{1t} can be determined in $\mathcal{O}(t)$ time for a fixed t, it follows from (5) that it takes $\mathcal{O}(T^2)$ time to evaluate $F(T)$. So the method proposed by Thomas is better than the heuristics proposed by Bhattacharjee and Ramesh in two ways. First, it is an exact algorithm instead of a heuristic approach. Second, the method appears to require a much lower running time. We implemented our algorithm in C++ and it takes a fraction of a second to solve a 1000-period problem instance, whereas Bhattacharjee and Ramesh only report results for problem instances up till 15 periods, since they compared the heuristics to the benchmark (optimal) approach which could only solve small instances. In this benchmark approach all possible production plans are enumerated, which are 2^{T-1} in total, for each of which a system of simultaneous non-linear equations needs to be solved. Independent of the time it takes to solve this system of equations, the exponential behavior of 2^{T-1} demonstrates the non-scalability of the benchmark approach. This is the reason for Bhattacharjee and Ramesh to develop heuristics as they mention (see p. 592): “The exponential nature of the structure of the problem and the characteristics of the maximum profit function precludes solution by using some of the standard available optimization techniques.” Indeed Bhattacharjee and Ramesh report computations times of more than one hour for optimally solving a 10-period problem instance, while a 15-period problem instance could not be solved within 4 hours of computation time. Even with the increase of computation power, it is not hard to verify that computation times will still be high even for relatively small instances.

Furthermore, we have implemented and tested the performance of the best performing heuristic in Bhattacharjee and Ramesh, called Algo II (the code is available on request). Unfortunately, we are not able to reproduce the results of Table 4 in their paper, because the unit purchase cost c is not specified. In fact, with the available parameter settings, it turns out that for low values of the purchase cost c, the optimal solution is trivial with procurement in each period, which is also found by our implementation of Algo II. On the other hand, for higher values of c, the optimal profit is negative and hence it is better not to serve any demand. Therefore, we have selected some other parameter settings for which the results are shown in Table 1. We report the average deviation from optimality over 100 instances, computed as $\frac{\Pi^* - \Pi^H}{\Pi^*}$ with Π^* (resp. Π^H) the optimal (resp. average heuristic) objective value. The table shows that for relatively small ordering cost K and small unit purchase cost c, Algo II yields an optimal solution. However, for some other parameter settings the deviation from optimality is over 100%. This means that $\Pi^H < 0 < \Pi^*$, that is, Algo II fails to find a profitable solution, while there exists one. This is a clear

Table 1
Average deviation from optimality (in %) of Algo II of Bhattacharjee and Ramesh. Other parameter settings: $\beta = 5000$, $h = 5$, $p_{\min} = 2$, $p_{\max} = \infty$, ITER = 30.

		$K = 1250$		$K = 1500$	
		$c = 0.75$	$c = 1.0$	$c = 0.75$	$c = 1.0$
$T = 5$	$\alpha = 1.5$	0.0	4.2	2.6	12.8
	$\alpha = 2.0$	0.0	59.0	33.5	349.0
$T = 10$	$\alpha = 1.5$	0.0	5.6	3.8	18.2
	$\alpha = 2.0$	0.0	66.7	35.4	418.2

reason why an optimal algorithm is preferred, besides the potential exponential running time of Algo II.

Finally, we have not explained yet how to take into account the restriction $p_{\min} \le p_t \le p_{\max}$ in our method. In turns out that this restriction does not make the problem harder to solve. Including this restriction, the price that maximizes (6) for each period i must be equal to $p_{\min}$, $p_{\max}$ or p_i^*. This means that we have to check a constant number of prices for determining the optimal profit of a subplan. So the (theoretical) running time of the algorithm is not affected by this restriction.

3.3. Improvement of the algorithmic running time

Instead of using recursion (5) to solve the problem, the solution procedure can be further improved. To see this, first note that given the valuesπ_{1t}^*, the problem can be considered as a partition problem. The problem is to partition the model horizon T into n subplans of length t_i $(i = 1, \ldots, n)$ each with a profit $\pi_{1t_i}^*$, such that the total profit of the subplans is maximized and $\sum_{i=1}^n t_i = T$. Orlin (1985) shows that the problem has a simple solution in case the function π_{1t}^* is concave.

Theorem 2. *The function π_{1t}^* is strictly increasing and concave in t.*

Proof. First, from (10) it follows that

$$\Delta\pi_t^* \equiv \pi_{1t}^* - \pi_{1,t-1}^* = (p_t^* - c - (t-1)h)\beta p_t^{*-\alpha}$$
$$= \beta\left(\frac{\alpha}{\alpha-1}\right)^{-\alpha}(c + (t-1)h)^{1-\alpha}.$$

It follows now immediately that $\Delta\pi_t^* > 0$, which means that $\Delta\pi_t^*$ is increasing in t. Furthermore, because $\Delta\pi_t^*$ is decreasing in t, π_{1t}^* is concave in t. □

The analysis of Orlin (1985) shows that if q is the subplan with maximum average profit per period, i.e., $q = \arg\max\{\frac{\pi_{1t}^*}{t} | t = 1, 2, \ldots\}$, then the horizon is partitioned into either $n = \lfloor \frac{T}{q} \rfloor$ or $n = \lceil \frac{T}{q} \rceil$ subplans. Furthermore, given n, each subplan is of length either $t = \lfloor \frac{T}{n} \rfloor$ or $t = \lceil \frac{T}{n} \rceil$. To be more precise, there are $n(t+1) - T$ subplans of length t and $T - tn$ subplans of length $t+1$. Because q is independent of T, this means that the problem can be solved in a time that is independent of T. However, because the input of the problem is of size $\mathcal{O}(\log(\max\{T, c, h, \alpha, \beta\})$, applying the approach of Orlin (1985) does not prove that the problem can be solved in polynomial time (or even in constant time). This is because the time to find the value q may not be polynomial in the problem parameters other than T.

It is not directly clear that π_{1t}^* is concave with the additional restriction $p_{\min} \le p_t \le p_{\max}$. The following theorem shows that the concavity property still holds and hence again Orlins approach can be applied.

Theorem 3. *The function π_{1t}^* is concave when the restriction $p_{\min} \le p_t \le p_{\max}$ holds.*

Proof. Let $\pi_i(p) = (p - c - (i-1)h)\beta p^{-\alpha}$, the marginal profit in period i when the price equals p. As already shown in Section 3.2, the price that maximizes $\pi_i(p)$ equals $p_i^* = \frac{\alpha(c+(i-1)h)}{\alpha-1}$ and $p_i^* < p_{i+1}^*$. Because $\pi_i(p)$ is unimodal in p, the price that maximizes $\pi_i(p)$ in case $p_i^* < p_{\min}$ equals $p_i = p_{\min}$ and the price that maximizes $\pi_i(p)$ in case $p_i^* > p_{\max}$ equals $p_i = p_{\max}$. So summarizing, the prices p_i that maximize π_{1t} equal

$$p_i = \begin{cases} p_{\min} & \text{if } p_i^* < p_{\min} \\ p_i^* & \text{if } p_{\min} \le p_i^* \le p_{\max} \\ p_{\max} & \text{if } p_i^* > p_{\max} \end{cases} \quad \text{for } i = 1, \ldots, t.$$

To show that π_{1t}^* is concave we have to show that $\pi_i(p_i)$ is decreasing in i. From Theorem 2 it follows that $\pi_i(p_i^*) \ge \pi_{i+1}(p_{i+1}^*)$. Furthermore, because $\pi_i(p) - \pi_{i+1}(p) = h\beta p^{-\alpha} > 0$, this also holds for $p_i = p_{i+1} = p_{\min}$ and $p_i = p_{i+1} = p_{\max}$. It remains to show that $\pi_i(p_i) - \pi_{i+1}(p_{i+1}) \ge 0$ for $p_i^* < p_{\min} \le p_{i+1}^*$ and $p_i^* \le p_{\max} < p_{i+1}^*$. If $p_i^* < p_{\min} \le p_{i+1}^*$, then $p_i = p_{\min}$, $p_{i+1} = p_i^*$ and hence

$$\pi_i(p_{\min}) \ge \pi_i(p_{i+1}^*) \ge \pi_{i+1}(p_{i+1}^*).$$

If $p_i^* \le p_{\max} < p_{i+1}^*$, then $p_i = p_i^*$, $p_{i+1} = p_{\max}$ and hence

$$\pi_i(p_i^*) \ge \pi_{i+1}(p_{i+1}^*) \ge \pi_{i+1}(p_{\max}).$$

Both cases are illustrated in Figs. 1 and 2. □

Finally, Bhattacharjee and Ramesh consider the case of perishable goods. They assume that the goods may perish after a fixed number of periods, say $m \le T$. It is also easy to extend Thomas' method with this additional feature. First, one can show that the optimal solution consists of consecutive subplans because of the constant cost parameters and therefore recursion (9) still holds. Second, it is never optimal to order for more than m periods, because this will lead to unnecessary purchasing and holding cost or

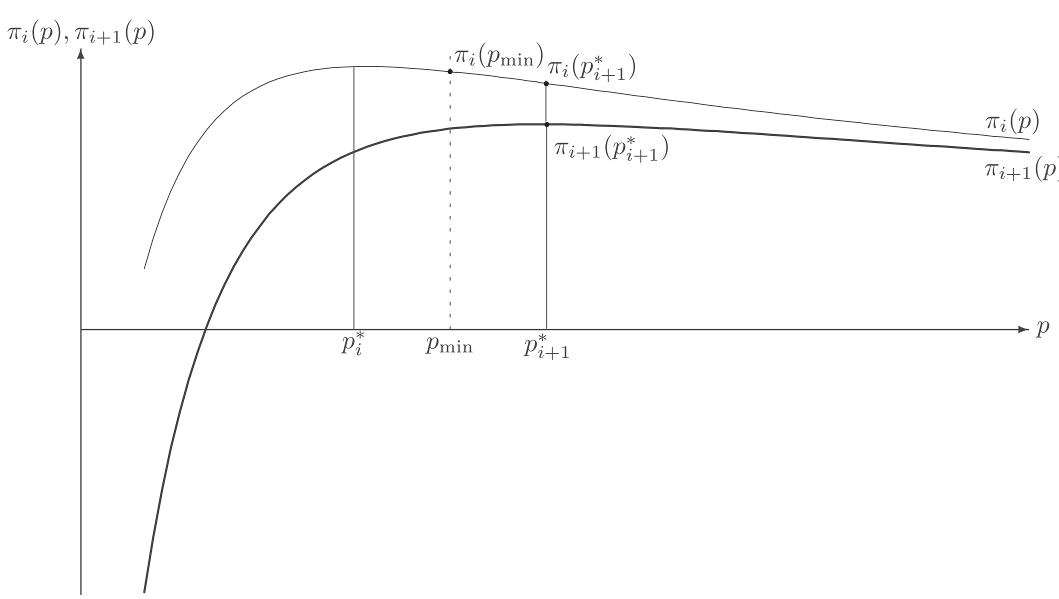

Fig. 1. Case $p_i^* < p_{\min} \le p_{i+1}^*$.

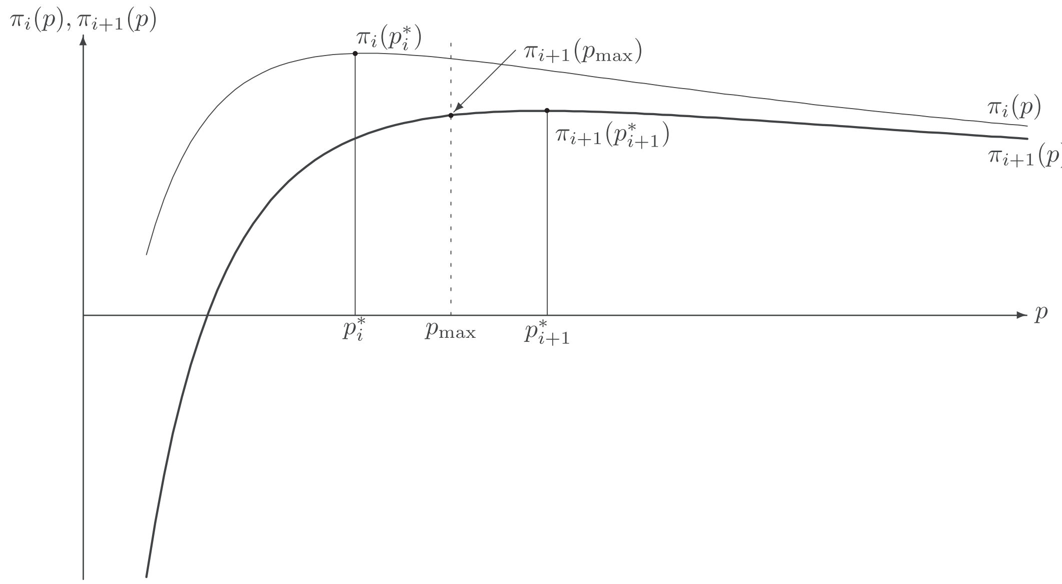

Fig. 2. Case $p_i^* \le p_{\max} < p_{i+1}^*$.

Table 2
Costs of Example 1.

t	1	2	3	4	5	6	7	8	9	10
p_t^*	4	6	8	10	12	14	16	18	20	22
π_{1t}^*	0	6.67	11.67	15.67	19.00	21.86	24.36	26.58	28.58	30.40
$\frac{\pi_{1t}^*}{t}$	0	3.33	3.89	3.92	3.80	3.64	3.48	3.32	3.18	3.04

unsatisfied demand. So for finding $F(t)$ in (5) we do not need to consider the term $F(j-1)+\pi_{jt}^*$ for all $j = 1,\dots,t$, but only for $j = \max\{1, t-m+1\},\dots,t$ and therefore only the values π_{1j}^* for $j = 1,\dots,m$ are needed. This means that in the case of perishable goods the running time of the algorithm reduces to $\mathcal{O}(mT)$.

Because we will never have subplans of more than m periods, it is not difficult to see that Orlins approach can also still be applied by setting $q := \min\{q, m\}$. What has to be taken into account is that it is not allowed to have subplans with more than m periods. This is not feasible because demands after the mth period of the subplan are not satisfied. The latter may occur when we have $n = \lfloor \frac{T}{q} \rfloor$ subplans with lengths of size $t = \lceil \frac{T}{n} \rceil$ and $t+1$. In this case the solution with $n = \lceil \frac{T}{q} \rceil$ is optimal. We illustrate Orlins approach in Example 1.

Example 1. Consider a problem instance with $K = 20$, $c = 2$, $h = 1$, $\alpha = 2$, $\beta = 80$ and $T = 10$, so the demand function in each period equals $d(p) = 80p^{-2}$. Using recursion formulas (9) and (10) the optimal prices and profit for a subplan of t periods can be calculated and are found in Table 2. It follows from Table 2 that the subplan with optimal average profit is of length 4, i.e., $q = 4$. This means that the optimal solution consists of partitioning the horizon in either $n = \lfloor \frac{T}{q} \rfloor = 2$ or $n = \lceil \frac{T}{q} \rceil = 3$ subplans.

- $n = 2$:
  There are only subplans of length $t = \lfloor \frac{T}{n} \rfloor = 5$ (as 2 exactly divides 10). The total revenu in this case equals $2\pi_{1,5}^* = 2 \cdot 19 = 38$.
- $n = 3$:
  There are subplans of length $t = \lfloor \frac{T}{n} \rfloor = 3$ and length $t = \lceil \frac{T}{n} \rceil = 4$. That is, we have two subplans of length 3 and one subplan of length 4 with total profit $2\pi_{1,3}^* + \pi_{1,4}^* = 2 \cdot 11.67 + 15.67 = 39$.

So the optimal solution is to partition the horizon into 2 subplans of length 3 and 1 of length 4 with a total profit of 39. Note that it is sufficient to calculate Table 2 up to $t = 5$, because at this point the optimal value q is found.

4. Concerns about the results presented in Bhattacharjee and Ramesh (2000)

Bhattacharjee and Ramesh proposed two heuristic algorithms to solve the joint pricing and planning problem. To justify the application of heuristics, they refer to the exponential nature of the problem and the characteristics of the maximum profit function. Indeed there are 2^{T-1} possible ordering policies (assuming positive demand in period 1). However, Thomas' approach (see Section 3) circumvents this 'exponentiality' and the problem can be solved to optimality by an efficient method. Note that the classical Wagner–Whitin problem also has an exponential number of possible ordering policies, but it can still be solved in polynomial time (see Wagner & Whitin, 1958).

Furthermore, in both heuristics there is a predetermined value r, which defines the maximum length of a subplan to be considered. For such a subplan all (2^{r-1}) ordering policies are generated at the start of the algorithms. This does not only mean that the heuristics have a running time which is exponential in r, but also that the heuristics may perform poorly if the optimal size of a subplan is larger than r. Bhattacharjee and Ramesh report worst case deviations of more than 28% and 18% for the two heuristics.

Besides the fact that the problem can be efficiently solved to optimality, Bhattacharjee and Ramesh also make several incorrect statements about the problem. Below we mention some of them.

1) Bhattacharjee and Ramesh (2000, Theorem 1, p. 587) claim that for a profit-maximizing firm it is always profitable to meet total demand. This means that shortage cost can be ignored and only the model with no loss of demand needs to be considered. In the proof of this result the authors use the fact that by increasing price in case of a shortage, there is an increase in revenue and a saving in shortage cost. However, later they assume that $p_{\min} \le p_t \le p_{\max}$. This is in contradiction with the proof where it is assumed that price can always be increased. In Example 2 we show that it can be optimal to have loss of demand.

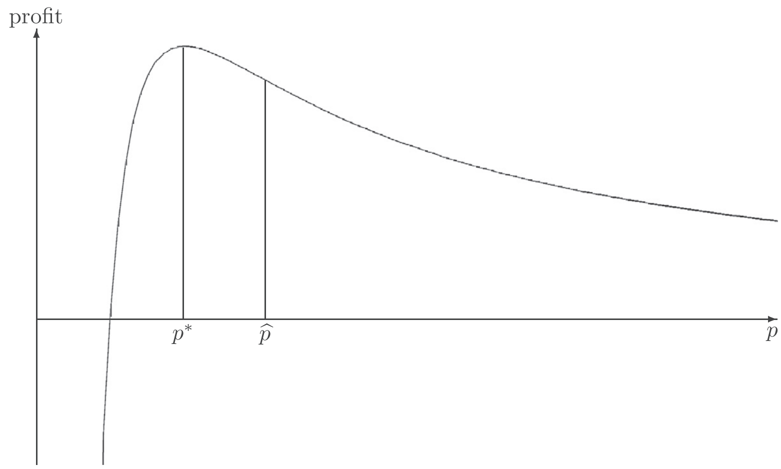

Fig. 3. Maximum profit function for a single period model.

Example 2. Consider a T-period example with $K = 9$, $p_{\max} - c = 3$, $p_{\min} - c > 0$, $s = 1$ and $h = 1$, where s is the per unit shortage cost. Furthermore, let the demand function be such that $d(p_{\max}) = 2$ and $(p_{\max} - c)d(p_{\max}) > (p - c)d(p)$ for all $p < p_{\max}$. For example, the function $d(p) = \frac{72}{p^2}$ with $p_{\min} = 4$, $p_{\max} = 6$ and $c = 3$ satisfies this property. Observe that

- it is not optimal to have both $p_t < p_{\max}$ and unsatisfied demand in that period t (see the argument used by Bhattacharjee and Ramesh in the "proof" of their Theorem 1)
- it is not optimal to have $p_t < p_{\max}$ and satisfy all demand in that period t (because it is better to set the price to $p_{\max}$ and then satisfy all demand)
- it is not optimal to have $p_t = p_{\max}$ and satisfy demand only partially in that period t (because also satisfying the remainder of the demand will increase profit and lower shortage costs).

It follows that in an optimal solution the price is set to $p_{\max}$ in every period and a period's demand is either completely satisfied or completely unsatisfied. One can easily verify that for the parameter values given above only production runs that cover the complete demand of two consecutive periods are profitable. A production run to cover only one period's demand is more expensive than leaving the demand unsatisfied. In case of three or more periods, the holding costs of the third period are higher than the shortage costs. Hence, the optimal solution is as follows. When T is even, there is a setup in every odd-numbered period to satisfy completely the 4 units of demand of that and the next period. In case T is odd, then there is exactly one (take any) odd-numbered period for which the 2 units of demand are not satisfied. For the remaining periods the solution has the same structure as in the case of an even number of periods.

2) The mathematical programming formulation of the problem presented in Bhattacharjee and Ramesh (2000, formula (2.2)–(2.6), p. 588) is incorrect. It is easily seen that it is always optimal (and feasible) in their formulation to take $q_t = 0$ for all periods. Of course, this is not a feasible solution to the 'real' problem where demand needs to be satisfied by positive production quantities. A correct formulation was presented in Section 2 of this note.

3) Bhattacharjee and Ramesh (2000, Theorem 2, p. 589) claim that the maximum profit function for a n-period subplan is concave. However, in Section 3 we showed that the maximum profit function is concave for $p < \widehat{p}$ and convex for $p > \widehat{p}$. A simple plot (see Fig. 3) of the profit function for a 1-period model (with $\alpha = 2$, $\beta = 3$, $K = 0.1$, $c = 1$ and $h = 1$) also shows this result.

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