Worst case analysis of Relax and Fix heuristics for lot-sizing problems

Nabil Absi[1] and Wilco van den Heuvel*[2]

[1]Ecole des Mines de Saint-Etienne and LIMOS, UMR CNRS 6158, CMP Georges Charpak, 13541 Gardanne, France

[2]Econometric Institute, Erasmus University Rotterdam, Rotterdam, The Netherlands

December 9, 2019

Abstract

We analyze the worst case behavior of Relax and Fix heuristics, which are often used to solve mixed integer linear programming problems. In these heuristics the integer variables are fixed in an iterative way, while having relaxed part of the integer variables in each iteration. In particular, we analyze Relax and Fix heuristics for lot-sizing problems. We show that even for simple instances with time-invariant parameters, the worst case ratio may be unbounded. Furthermore, we show some counterintuitive behavior of Relax and Fix heuristics from computational experiments.

Keywords: Production; lot-sizing; relax and fix heuristics; worst case analysis

1 Introduction

In the last fifteen years we have experienced an increasing number of industries that try to optimize their supply chain planning using off-the-shelf Mixed Integer Programming (MIP) solvers to deal with mixed integer models. When the corresponding models become large and harder to solve in a reasonable computation time, a natural way to obtain good feasible

*corresponding author

solutions is to use MIP-based heuristics. The general purpose of these heuristics is to decompose the monolithic model into easier to solve sub-models in order to obtain good feasible solutions. These sub-models are subsequently solved using a given scheme so as to construct a good feasible solution for the original model. The effectiveness of these MIP-based heuristics highly depends on the way the problem is decomposed.

To solve lot-sizing problems, several MIP-based approaches were proposed in the literature. A natural and often used method is the so-called Relax and Fix heuristic. The general idea of this heuristic is described by the following steps. (1) Maintain integrality on a set of integer decision variables that are not fixed yet, relax the remaining unfixed integer decision variables and solve the resulting problem. (2) Fix part of the integer variables that were not relaxed and not fixed yet according to the obtained solution. (3) Repeat Steps 1 and 2 until all integer variables are fixed or when the obtained solution is integer feasible.

This approach is often applied in a rolling horizon fashion where a time window (interval of consecutive periods), in which integrality constraints on decision variables are maintained, is moved in a forward way. The integer decision variables (or a part of them) are fixed, after which the time window is moved forward. Typically, these time windows may overlap in order to improve solution quality.

Recent research has shown that this approach performs well on average, by testing it on industrial instances or sets of randomly generated instances for several types of lot-sizing problems. However, to the best of our knowledge, no research has been done on the worst case behavior of these Relax and Fix heuristics. The goal of this paper is to conduct a first study that presents a worst case analysis of Relax and Fix heuristics for lot-sizing problems. Two main research questions are addressed. What is the worst case behavior of Relax and Fix heuristics? Can we get more insight in the behavior of Relax and Fix heuristics?

The rest of the paper is organized as follows. In Section 2, we first present a general framework of Relax and Fix heuristics, followed by a literature review of Relax and Fix and closely related heuristics that are applied to lot-sizing problems. In Section 3 we analyze a Relax and Fix heuristic for the classical uncapacitated lot-sizing problem. In Section 4, we present computational experiments which show some counterintuitive results. Finally, we close our paper with discussions and some future research directions.

2 Relax and Fix heuristics for lot-sizing problems

2.1 A general framework for the Relax and Fix heuristic

In what follows we present a general scheme of a Relax and Fix heuristic applied to a generic mixed integer program (see also Wolsey (1998)). Consider the following general MIP model defined on two sets of decision variables: continuous variables $x_k,\ k \in K$ and integer variables $y_j,\ j \in J$. The model consists of m constraints, and the problem parameters are denoted by c_k, d_j, a_{ik}, e_{ij} and b_i. The generic MIP model is defined as follows:

$$\begin{array}{llll} \min & \sum_{k\in K} c_k x_k + \sum_{j \in J} d_j y_j & & \\ \text{s.t.} & \sum_{k\in K} a_{ik} x_k + \sum_{j\in J} e_{ij} y_j & \leq \quad b_i & 1 \leq i \leq m \\ & x_k \geq 0 & & k \in K \\ & y_j \in \mathbb{N} & & j \in J. \end{array}$$

The Relax and Fix method is an iterative method, where the subproblem to be solved in iteration k is defined by the following sets, which are specified by the user:

- $I_k \subset J$ the subset of integer variables that will be optimized over in iteration k,
- $F_k \subseteq I_k$ the set of integer variables that will be fixed

For convenience, we define $\bar{I}_k = \bigcup_{i=1,\ldots,k} I_i$ (resp. $\bar{F}_k = \bigcup_{i=1,\ldots,k} F_i$) as the set of variables which has been optimized over (resp. has been fixed) up till iteration k. A general Relax and Fix heuristic works as follows:

Step 0 set $F_0 = \emptyset$ and $k = 1$

Step 1 construct problem (P_k), such that the variables:

- y_j are fixed for $j \in \bar{F}_{k-1}$,

- y_j are integer for $j \in I_k$, and

- y_j are relaxed (to continuous variables) for $j \in J - \bar{I}_k$

Step 2 solve (P_k):

- fix the integer variables y_j with $j \in F_k$ to the obtained optimal values

Step 3 while not all variables are fixed and the solution is not integer:

- set $k = k + 1$
- go to Step 1

Figure 1 provides an illustration of three iterations of a Relax and Fix heuristic. Just above a horizontal line it is indicated which variables are optimized over and which variables are relaxed, while just below a horizontal line it is specified which variables were fixed in earlier iterations and are fixed in the current iteration.

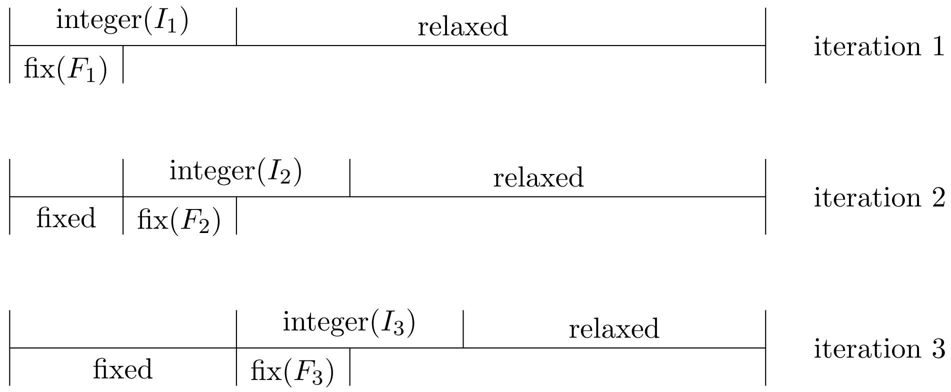

Figure 1: Illustration of the Relax and Fix heuristic

2.2 Literature review

Before the nineties, the most used heuristic approaches to solve lot-sizing problems were either based on Lagrangian relaxation (see, e.g., Trigeiro et al. (1989) for multi-item lot-sizing problems), or based on "intuitive" rules coming from the structural properties holding for the single-item lot-sizing problem such as the Economic order quantity, Part period balancing, Silver and Meal, and Least cost per unit rule (Baker, 1989). Since the beginning of the nineties, several heuristics based on Linear Programming (LP-based heuristics) were proposed to solve several MIP models. For example, Maes et al. (1991) proposed approaches based on rounding an LP solution for the multi-level capacitated lot-sizing problem. Alfieri et al. (2002) use LP-based heuristics to solve the classical capacitated multi-item lot-sizing problem.

With the emergence of off-the-shelf MIP-solvers in the late nineties, MIP-based heuristics started to be used to solve MIP models. For example, Belvaux et al. (1998) applied such a method (called Relax and Fix heuristic) to the optimal placement of multiplexers. This method is presented in a generic way in Wolsey (1998). Note that the class of MIP-based heuristics is more general than the class of Relax and Fix heuristics, and we mainly review the latter class and some heurictics which are closely related.

The first applications of MIP-based heuristics to solve lot-sizing problems started in the beginning of 2000. For example, Clark and Clark (2000) applied a Relax and Fix heuristic to solve the capacitated multi-item lot-sizing problem with sequence dependent setup times. This heuristic was specifically designed to be applied on a rolling horizon basis. Later, several studies (Mercé and Fontan (2003), Stadtler (2003), Suerie and Stadtler (2003), Clark (2003), Federgruen et al. (2007), Absi and Kedad-Sidhoum (2007), de Araujo et al. (2007), Akartunalı and Miller (2009)) used similar approaches to solve different variants of the lot-sizing problem (classical or industrial). Pochet and Van Vyve (2004) compared several heuristics for production planning problems in a computational experiment: different versions of LP-based heuristics (Successive Rounding, Probabilistic Fixing, Successive Probabilistic Fixing), Truncated Branch and Bound, Relax and Fix, and an Iterative Production Estimate Heuristic. According to their experiments, the Relax and Fix heuristic performs well on easy and medium problems, while the Iterative Production Estimate Heuristic performs well on average.

Recently, an alternative version of the Relax and Fix heuristic was proposed in the literature under the name of Fix and Optimize (Helber and Sahling, 2010). Contrary to the Relax and Fix heuristic applied on a rolling horizon basis, in the Fix and Optimize heuristic the set of binary variables is decomposed into fixed variables and variables being optimized over, where fixed variables maybe unfixed again. This decomposition can, for example, be product oriented, resource oriented or process oriented. Note that the Fix and Optimize heuristic can be used to improve a feasible solution obtained by a rolling horizon-based Relax and Fix heuristic. To deal with the multi-level lot-sizing problem, Helber and Sahling (2010) show empirically that the Fix and Optimize heuristic outperforms the rolling horizon-based Relax and Fix heuristic.

Table 1 presents a non-exhaustive list of contributions where Relax and Fix heuristics are applied to solve a lot-sizing problem. The particularity of each contribution is highlighted by a cross. In the columns “single(s)/multi res.(m)”, “s” means a single resource while “m” means multiple resources. In the column “formulation”, “AGG” means the straightforward aggregated formulation is used.

Authors	**year**	**multi-level**	**single (s)/multiple (m) resources**	**setup times**	**sequence dependent /changeover**	**setup carryover**	**backlogs**	**lost sales**	**overtime**	**production/inventory capacity**	**formulation**
Belvaux and Wolsey	2000	x	m	x	x		x		x	x	AGG
Mercé and Fontan	2003		m	x			x			x	AGG
Stadtler	2003	x	m	x					x	x	AGG
Suerie and Stadtler	2003		m	x		x				x	AGG
Pochet and Van Vyve	2004	x	s/m	x			x		x	x	AGG
Absi and Kedad-Sidhoum	2007		m	x				x		x	AGG
Federgruen et al.	2007		s							x	AGG
de Araujo et al.	2007	x	m	x	x		x			x	AGG
Akartunalı and Miller	2009	x	m	x						x	AGG
Mohammadi et al.	2010	x	m	x	x					x	AGG
James and Almada-Lobo	2011		m	x		x				x	AGG
Goren et al.	2012		s	x	x	x				x	AGG
Santos and Almada-Lobo	2012		s		x		x			x	AGG
Wu et al.	2012	x	m	x					x	x	AGG
Ramezanian and Saidi-Mehrabad	2013		m		x		x				AGG
Sel and Bilgen	2014	x	m	x			x			x	AGG
Toledo et al.	2015	x	m	x			x			x	AGG
Chen	2015	x	m		x				x	x	AGG
Tempelmeier and Copil	2016		m	x		x				x	AGG
Aouam et al.	2018		s				x			x	AGG

Table 1: Relax and Fix heuristic: a non exhaustive list of contributions

From Table 1, we notice that Relax and Fix heuristics are mainly used to solve complicated lot-sizing problems (like multi-level, multiple resources, setup times, etc.). We also notice that all studies considered use the straightforward aggregated formulation. This is mainly justified by the huge number of continuous variables that are used in disaggregated formulations like the facility location formulation (Krarup and Bilde, 1977) or shortest path formulation (Eppen and Martin, 1987). However, in the studies of Belvaux and Wolsey (2000) and Akartunalı and Miller (2009) the formulations are strengthened with valid inequalities. The authors use generalized versions of the (l, S) inequalities introduced by Barany et al. (1984). Note that these (l, S) inequalities define the convex hull of the single-item uncapacitated lot-sizing problem. The number of these inequalities is exponential, but separation can be done in polynomial time. Belvaux and Wolsey (2000) proposed a prototype modelling and optimization system called *bc-prod* to tackle a variety of lot-sizing problems (multiple items, multiple machines, multiple levels, small and big time buckets, etc.). *bc-prod* is based on a commercial solver (XPRESS-MP) and includes the generation of cutting planes, after which a Relax and Fix heuristic can be applied. Akartunalı and Miller (2009) propose a Relax and Fix based heuristic for the multi-level lot-sizing problem with multiple items and multiple machines. In their approach also (l, S) inequalities are used to strengthen the aggregate formulation.

In the literature, several articles deal with studying the performance of "intuitive" heuristics for the classical uncapacitated single-item lot-sizing problem. We can cite for instance Axsäter (1982), Bitran et al. (1984), Axsäter (1985), Vachani (1992) who provide worst case analysis for some of these heuristics, while Baker (1989) tests a large number of these heuristics in a computational study. Van den Heuvel and Wagelmans (2010) analyzed the worst case performance of heuristics for the classical lot-sizing problem. They consider a general class of online heuristics that is generally used in a rolling horizon environment and show that any online heuristic has a worst case ratio of at best 2.

This literature review shows that Relax and Fix heuristics are often used to solve lot-sizing problems, but their worst case analysis has not been studied before. It also shows that the majority of studies use the classical aggregate formulation. This motivates to study the performance of the Relax and Fix heuristic on this type of formulation.

Finally, we like to mention that Relax and Fix heuristics are not only applied to lot-sizing problems. Recent studies show that this method is applied to a variety of problems such as: maritime inventory routing (dos Santos Diz et al., 2018), operating rooms scheduling (Kroer et al., 2018), harvest scheduling (Junqueira and Morabito, 2019) and vehicle-reservation assignment (Oliveira et al., 2014).

3 Analyzing Relax and Fix heuristics for the ULS problem

3.1 Preliminaries

In this section, we conduct a worst case analysis of Relax and Fix heuristics on the classical uncapacitated single-item lot-sizing problem (ULS). Note that ULS is a special case of all models mentioned in the overview of Table 1. This means that any worst case bound derived for the ULS problem is a lower bound on the worst case bound of the other models. Note that this statement does not hold in case the aggregate formulation is strengthened with valid inequalities, as done by Belvaux and Wolsey (2000) and Akartunalı and Miller (2009). In particular, if the worst case bound under a particular setting of the Relax and Fix heuristic is unbounded for the ULS problem, then there is no need to analyze the worst case bound of the capacitated version of the problem under this setting, as it will be unbounded as well.

We first analyze the general ULS case with a time window of length 1 in Section 3.2, in which we derive properties that eases the analysis of special cases in Section 3.3. Then we provide worst case results of Relax and Fix heuristics under several settings. We start with a case (constant costs and large big M) that is relatively easy to analyze and which turns out to be related to a heuristic from the literature (Section 3.3.1). Then we move to settings of the Relax and Fix heuristic that are more realistic (tighter big M and time window lengths of more than 1 period) but for which the instance parameters are still restricted (constant costs and demands; Section 3.3.2). Again, note that any worst case bound for restricted instance parameters, implies a bound that is at least as high for the case with more general parameters. From that perspective, one prefers to find bounds for problem instances where the parameters are as restricted as possible.

The classical ULS problem is defined on a discrete time horizon of T periods. Each period $t \in \{1, \ldots, T\}$ is characterized by a demand d_t to be satisfied and three cost components: f_t is a setup cost that is incurred if production takes place at period t, c_t is the unitary production cost that is counted for each produced unit at period t, and h_t is the cost of storing a unit from period t to period $t+1$. In the ULS problem one should decide when and how much to produce and how much to store in order to satisfy demands while minimizing the total production and holding costs. Backlogging and lost sales are not allowed. The classical straightforward aggregated formulation is defined by three sets of decision variables:

x_t the quantity to be produced in period t, y_t the binary setup variable that is equal to 1 if production takes place at period t (i.e., $x_t > 0$) and 0 otherwise, and I_t the inventory level at the end of period t. The ULS problem is formulated as follows (for convenience we use $[a, b]$ as a shorthand notation for $\{a, a+1, \ldots, b\}$).

$$\min \quad \sum_{t=1}^{T} (f_t y_t + c_t x_t + h_t I_t) \tag{1}$$

$$\text{s.t.} \quad I_t = I_{t-1} + x_t - d_t \qquad t \in [1, T] \tag{2}$$

$$x_t \leq M_t y_t \qquad t \in [1, T] \tag{3}$$

$$y_t \in \{0, 1\} \qquad t \in [1, T] \tag{4}$$

$$x_t, I_t \geq 0 \qquad t \in [1, T]. \tag{5}$$

The objective function (1) minimizes the sum of the setup, production and inventory holding costs. Constraints (2) are the inventory balance equations. Constraints (3) relate the continuous production variables to the binary setup variables. Constraints (4) and (5) specify the domains of the decision variables.

3.2 The general case with time window length 1

We start with analyzing the simplest case: an uncapacitated single-item lot-sizing (ULS) using the aggregate formulation together with a time window of length 1, i.e., $I_k = F_k = \{k\}$ for $k = 1, \ldots, T$. We will focus on iteration k of the Relax and Fix heuristic. That is, the binary variables are fixed in $[1, k-1]$, the binary variables are relaxed in $[k+1, T]$, while we optimize over the binary variable in period k.

Let $j < k$ be the last period with $y_j^* = 1$. As we will prove later, the Zero Inventory Ordering (ZIO) policy holds in the interval $[1, j-1]$ and hence decisions in this interval do not affect decisions in period k or further periods. (Recall that the ZIO property states that production can only occur when incoming inventory is zero, i.e., $I_{t-1} x_t = 0$ for each period t.) Therefore, we assume w.l.o.g. that $y_1^* = 1$ and $y_t^* = 0$ for $t = 2, \ldots, k-1$ in the analysis. Then, in iteration k we need to solve the following relaxed ULS problem:

$$\min \quad \sum_{t=1}^{T} (f_t y_t + c_t x_t + h_t I_t) \tag{6}$$

$$\text{s.t.} \quad I_t = I_{t-1} + x_t - d_t \qquad t \in [1, T] \tag{7}$$

$$x_t \leq M_t y_t \qquad t \in [k, T] \tag{8}$$

$$y_1 = 1 \tag{9}$$

$$y_t = 0 \qquad t \in [2, k-1] \tag{10}$$

$$y_k \in \{0, 1\} \tag{11}$$

$$0 \leq y_t \leq 1 \qquad t \in [k+1, T]. \tag{12}$$

Note that the above problem is a Mixed Integer program (MIP) with a single binary variable and hence we can solve two LP problems instead. Let $C^0_{LP}(k)$ (resp. $C^1_{LP}(k)$) be the optimal cost of the LP with $y_k = 0$ (resp. $y_k = 1$). Then, a new setup will be placed in period k if $C^1_{LP}(k) \leq C^0_{LP}(k)$. We will now show that we can find a closed form expression for $C^0_{LP}(k)$ and $C^1_{LP}(k)$.

First of all, because variables y_t ($t \in [k+1, T]$) only appear in (8) and the objective, it is optimal to set these variables as small as possible and hence (8) must be tight implying that $y_t = \frac{x_t}{M_t}$. After having performed this substitution, it follows that there is an extra variable production cost term $\frac{f_t}{M_t}$ in addition to c_t. Finally, the problem left can be considered as a network flow problem with linear costs and a single source. An illustration of the network corresponding to $C^0_{LP}(k)$ can be found in Figure 2.

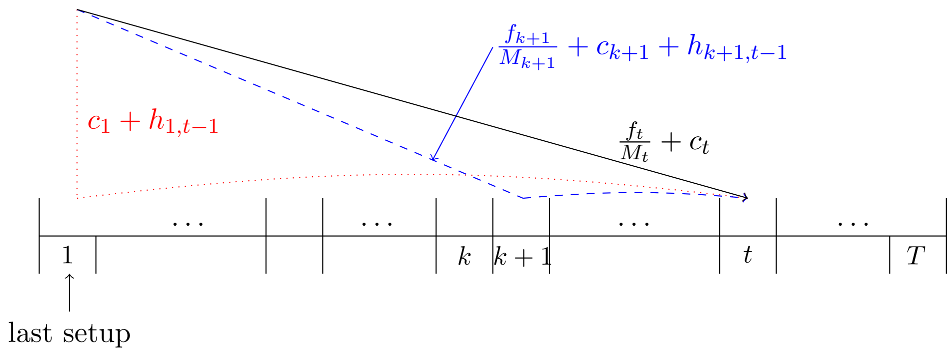

Figure 2: Network flow problem corresponding to $C^0_{LP}(k)$

As the network has a single source, we need to find the shortest path from the source node to node t, in order to satisfy the demand in some period $t \in [k+1, T]$. Moreover, the demand quantity itself does not effect the shortest path. In other words, we can consider each demand period t separately in order to determine $C^0_{LP}(k)$ and $C^1_{LP}(k)$.

We now focus on computing the cost to satisfy the demand in some period $t \in [k+1, T]$. First of all, assume that demand in period t is satisfied by production from the interval

$[k+1,t]$. Then the best production period j is found by

$$\min_{j\in[k+1,t]}\left\{\frac{f_j}{M_j}+c_j+h_{j,t-1}\right\}, \tag{13}$$

where $h_{j,k}=\sum_{i=j}^{k} h_i$. In case of $y_k=0$ (so when computing $C^0_{LP}(k)$), we can also satisfy this demand from period 1 incurring a unit cost of $c_1+h_{1,t-1}$. Hence, in case $y_k=0$ the optimal unit cost denoted by c^0_t equals

$$c^0_t=\min\left\{c_1+h_{1,t-1},\min_{j\in[k+1,t]}\left\{\frac{f_j}{M_j}+c_j+h_{j,t-1}\right\}\right\}. \tag{14}$$

Note that c^0_t depends on k, but for convenience we omit this dependence in the notation. Similarly, in case $y_k=1$ (so when computing $C^1_{LP}(k)$) the optimal unit cost, denoted by c^1_t, equals

$$c^1_t=\min\left\{c_1+h_{1,t-1},c_k+h_{k,t-1},\min_{j\in[k+1,t]}\left\{\frac{f_j}{M_j}+c_j+h_{j,t-1}\right\}\right\}. \tag{15}$$

Combining the above and adding the cost terms in the interval $[1,k]$ leads to the expressions

$$C^0_{LP}(k)=f_1+\sum_{t=1}^{k}(c_1+h_{1,t-1})d_t+\sum_{t=k+1}^{T}c^0_t d_t$$

and

$$C^1_{LP}(k)=f_1+\sum_{t=1}^{k-1}(c_1+h_{1,t-1})d_t+f_k+c_k d_k+\sum_{t=k+1}^{T}c^1_t d_t.$$

In order to analyze the Relax and Fix heuristic further in the upcoming sections, we define

$$\Delta C(k)=C^1_{LP}(k)-C^0_{LP}(k).$$

Clearly, the first period $k>1$ for which $\Delta C(k)\le 0$, will be the next setup period in the Relax and Fix heuristic (note that we can also use the rule $\Delta C(k)<0$ instead). The following property shows that the Relax and Fix heuristic generates consecutive setup periods in which the marginal costs are non-increasing over time.

Property 1 *Let $k>1$ be the first (non-trivial) setup generated by the Relax and Fix heuristic. Then $c_k\le c_1+h_{1,k-1}$.*

Proof By contradiction, assume that $\Delta C(k)\le 0$ and $c_k>c_1+h_{1,k-1}$. Note that the latter implies $c_k+h_{k,t-1}>c_1+h_{1,t-1}$. In turn, this implies that the minimum in (15) does not occur at the term $c_k+h_{k,t-1}$ and hence $c^1_0=c^1_t$. But then

$$\Delta C(k)=f_k+c_k d_k-(c_1+h_{1,k-1})d_k=f_k+(c_k-(c_1+h_{1,k-1}))d_k>0,$$

which is a contradiction with $\Delta C(k)\le 0$. □

Note that Property 1 implies that any demand will always be satisfied from the most recent setup period, which means that the ZIO property holds as claimed earlier. In order to compute $\Delta C(k)$ efficiently, it is convenient to use the minimizers in (14) (resp. (15)), which we denote by j_t^0 (resp. j_t^1). To ensure a unique minimizer, we choose the largest period j which minimizes (14) (resp. (15)). In particular, if the minimum takes place in the first term of (14) (resp. second term of (15)), then we let $j_t^0 = 1$ (resp. $j_t^1 = k$). We now present two properties to derive a corollary that will be used to ease the analysis in the subsequent sections.

Property 2 *For $t \in [k+1, T-1]$ we have $j_t^0 \leq j_{t+1}^0$ and $j_t^1 \leq j_{t+1}^1$.*

Proof When comparing the terms in the minimization for c_t^0 and c_{t+1}^0 in (14), we observe that (i) each term in c_t^0 changes by the same constant h_t, and (ii) there is one new term in c_{t+1}^0 corresponding to index $t+1$, which is the largest of all indices. As j_t^0 is defined as the largest index for which the minimum occurs, we either have that $j_{t+1}^0 = j_t^0$ (in case the minimum does not occur at the new term) or $j_{t+1}^0 = t+1$ (in case the minimum does not occur at the new term), implying that $j_{t+1}^0 \geq j_t^0$. A similar argument holds for j_t^1. □

Property 3 *If $j_{\hat{t}}^1 > k$ for some $\hat{t} > k$, then $j_t^0 = j_t^1$ for each $t \in [\hat{t}, T]$.*

Proof By Property 2, we have $j_t^1 \geq j_{\hat{t}}^1 > k$ for $t \in [\hat{t}, T]$. Because (i) all terms in (14) and (15) are equivalent except for the extra term $c_k + h_{k,t-1}$ in (15), (ii) the minimum does not occur at the extra term (as $j_t^1 > k$), and (iii) the largest minimizer is chosen in j_t^0 and j_t^1, it follows that $j_t^0 = j_t^1$. □

As a direct consequence of Property 3, we have the following corollary, which will be used to ease the analysis in the next sections.

Corollary 4 (Cost truncation) *If $j_{\hat{t}}^1 > k$ for some $\hat{t} > k$, then the cost terms corresponding to demand periods $[\hat{t}, T]$ can be ignored when computing $\Delta C(k)$.*

Proof Because $j_{\hat{t}}^1 > k$, we have from Property 3 that $j_t^0 = j_t^1$ for $t \in [\hat{t}, T]$. Furthermore, because $j_t^0 = j_t^1$ implies $c_t^0 = c_t^1$, the terms corresponding to demands in $[\hat{t}, T]$ cancel out when computing $C_{LP}^1(k) - C_{LP}^0(k)$ in $\Delta C(k)$ and the result follows. □

3.3 Worst case results under several settings

In this section we analyze the worst case ratio of the Relax and Fix heuristic under several settings and assumptions on the problem instance parameters. We assume in each case that we are in iteration k and need to determine whether to place a setup in period k.

3.3.1 Constant production costs and M large

Consider a problem instance with $f_t = f$, $c_t = c$ (and hence $c = 0$ w.l.o.g.), $h_t > 0$ and $M_t = M$ is sufficiently large. It is not difficult to see that for each demand period $t \in [k+1, T]$ the optimal production period is period t itself (i.e., $j_t^0 = j_t^1 = t$) and the unit cost equals becomes $c_t^0 = c_t^1 = \frac{f}{M}$. Corollary 4 implies that the Relax and Fix heuristic will not use any future demand information after period k when determining the next setup period. In particular, it follows that $C(k)$ reduces to the simple expression

$$\Delta C(k) = f - h_{1,k-1} d_k.$$

In other words, the next setup will occur in the first period $k > 1$ for which

$$h_{1,k-1} d_k \geq f,$$

or in words, when the holding costs of the demand exceeds the setup cost. This heuristic is also known as the Freeland-Colley heuristic (Freeland and Colley, 1982), and this heuristic has an unbounded worst-case ratio (see e.g. Vachani (1992)). This leads to the following proposition.

Proposition 5 *The worst-case ratio of the Relax and Fix heuristic with a large big M in the ULS formulation is unbounded.*

3.3.2 Constant costs and demands

One may argue that the reason for the bad performance is because of the big M that is chosen poorly. Therefore, we now consider an instance where the big M is chosen as tightly as possible in terms of the demand. Consider the problem instance with $f_t = f = T > 2$, $c_t = c = 0$, $h_t = h = 1$ and $d_t = d = 1$. We set M_t as small as possible, that is, $M_t = \sum_{i=t}^{T} d_i = T - t + 1$ being equal to the remaining demand.

Time window length one We will show that the first setup period will not occur before period $\frac{1}{2}T$. Assume that we are at iteration k (corresponding to the setup decision of period k) with $1 < k < \frac{1}{2}T$ and consider period $t = k + 2$. For the given parameters, we find that

$$\begin{aligned} c^1_{k+2} &= \min\left\{t-k, \min_{j\in\{k+1,k+2\}}\left\{\frac{f}{T-j+1}+t-j\right\}\right\} \\ &= \min\left\{2, \frac{f}{T-k}+1, \frac{f}{T-(k+1)}\right\} = \frac{f}{T-(k+1)}. \end{aligned}$$

This follows from the fact that $1 < \frac{f}{T-i} \leq 2$ for $1 < i \leq \frac{1}{2}T$. This means that $j^1_{k+2} = k+2$, and using Corollary 4, the only terms that matter to compute $\Delta C(k)$ are the ones up to period $t = k+1$. Summing up these terms up till this period we get the truncated costs

$$\tilde{C}^0_{LP}(k) = f + \tfrac{1}{2}k(k-1) + \frac{f}{T-k}$$

and

$$\tilde{C}^1_{LP}(k) = f + \tfrac{1}{2}(k-1)(k-2) + f + 1,$$

resulting in

$$\Delta C(k) = f + 2 - k - \frac{f}{T-k}.$$

It is not difficult to see that $\Delta C(k) > 0$ for $k < \frac{1}{2}T$ and $f = T$. Therefore, the next setup occurs in some period $k \geq \frac{1}{2}T$. This means that the holding cost in the first $\frac{1}{2}T - 1$ periods is $\frac{1}{2}(\frac{1}{2}T-1)(\frac{1}{2}T-2)$. In turn, this implies that the cost of the Relax and Fix heuristic is of size $\Omega(T^2)$.

On the other hand the Economic Order Quantity (EOQ) equals $\sqrt{(2fd)/h} = \sqrt{2T}$. Hence the cost in each order interval is approximately $T + \frac{1}{2}\sqrt{2T}(\sqrt{2T}-1) \in \mathcal{O}(T)$. Since we have $T/\sqrt{2T}$ intervals, the total cost of an optimal solution is $\mathcal{O}(T\sqrt{T})$. Hence, the worst case ratio will be arbitrarily large for T sufficiently large. This leads to the following proposition.

Proposition 6 *In case of a time window of length one, the worst-case ratio of the Relax and Fix heuristic with tight big M in the ULS formulation (in terms of demand) is unbounded.*

Time window length two The analysis will be similar to the time window length one case. We will again show that the first setup period will not occur before period $\frac{1}{2}T$. Assume that we are at iteration k with $1 < k < \frac{1}{2}T - 1$. Then we need to find the best solution among the ones with either or not a setup in period k and $k+1$. Let c^{ab}_t (resp. j^{ab}_t) with $a, b \in \{0, 1\}$

be the corresponding 'extended' notation (i.e., applying to the time window length two case). When considering period $t = k+3$, we find that

$$c_{k+3}^{11} = \min\left\{2, \frac{f}{T-(k+1)} + 1, \frac{f}{T-(k+2)}\right\} = \frac{f}{T-(k+2)}.$$

This means that $j_{k+3}^{ab} = k+3$ for all $a, b \in \{0,1\}$. Using a generalization of Corollary 4, the only terms that matter to find the best solution among the four are the ones up to period $t = k+2$. Summing up these terms we find

$$\begin{aligned}
\tilde{C}_{LP}^{00}(k) &= f + \tfrac{1}{2}k(k+1) + \frac{f}{T-k-1} \\
\tilde{C}_{LP}^{01}(k) &= f + \tfrac{1}{2}k(k-1) + f + 1 \\
\tilde{C}_{LP}^{10}(k) &= f + \tfrac{1}{2}(k-1)(k-2) + f + 1 + \frac{f}{T-k-1} \\
\tilde{C}_{LP}^{11}(k) &= f + \tfrac{1}{2}(k-1)(k-2) + f + f + 1.
\end{aligned}$$

Furthermore, using that $f = T$ and $k < \frac{1}{2}T - 1$ we find that

$$\begin{aligned}
\tilde{C}_{LP}^{01}(k) - \tilde{C}_{LP}^{00}(k) &= f + 1 - k - \frac{f}{T-k-1} > 0 \\
\tilde{C}_{LP}^{10}(k) - \tilde{C}_{LP}^{00}(k) &= f - 2(k-1) > 0 \\
\tilde{C}_{LP}^{11}(k) - \tilde{C}_{LP}^{10}(k) &= f - \frac{f}{T-k-1} > 0.
\end{aligned}$$

This means that $\tilde{C}_{LP}^{00}(k)$ is the minimum and the Relax and Fix heuristic will generate no setups in periods k and $k+1$. Therefore, as in the time window length one case, the next setup period occurs in some period $k \geq \frac{1}{2}T$. This leads to the following proposition.

Proposition 7 *In case of a time window of length two, the worst-case ratio of the Relax and Fix heuristic with tight big M (in terms of demand) is unbounded.*

General time window length Suppose now that we have a general but fixed time window of length $\ell \geq 3$ but with $\ell \leq \sqrt{T}$. It turns out that the latter bound on ℓ is needed in the analysis to arrive at the main result of this section. Suppose that we performed some iterations and assume that we are in the following situation: setups in $[1, k-\ell]$ with $2\ell \leq k \leq \frac{1}{2}T$ are fixed with only a setup in period 1, the variables in $[k-\ell+1, k]$ are binary, and the variables in $[k+1, T]$ are relaxed. We will solve this MIP problem by inspection.

First, we determine which are the relevant periods to consider in the optimization problem. To that end, it is convenient to define

$$c_t^p = \min\left\{t - p, \min_{j\in[k+1,p]}\left\{\frac{f}{T-j+1} + t - j\right\}\right\}, \tag{16}$$

that is, the minimum variable cost to satisfy demand in period t when period p is the last setup (again, we omit k for notional convenience). One can verify that $c_{k+1}^p = 1$ for $p = k$ and $c_{k+1}^p = \frac{f}{T-k}$ for $p < k$. Furthermore, $c_{k+2}^p = \frac{f}{T-(k+1)}$ for any $p \leq k$. Note that $1 < \frac{f}{T-k+1} < 2$ when $k \leq \frac{1}{2}T$. This implies that for any two solutions of the MIP (i) only the cost to satisfy demand in period $k+1$ may be different, and (ii) this cost difference is at most 1 (namely at most $\frac{f}{T-k} - 1$). Because the setup cost $f = T$ is much larger than 1, the number of setups will be the determining factor in the optimization, and we only need to consider the cost in interval $[k-\ell+1, k]$.

Now let us focus on the optimal number of setups in $[k-\ell+1, k]$. First, suppose that there are at least two setups, say in periods $p_1, p_2 \in [k-\ell+1, k]$ with $p_1 < p_2$. When removing the setup in period p_2 one can verify that the cost saving is at least

$$f + \sum_{j=p_2}^{k}(j - p_2) - \sum_{j=p_2}^{k}(j - p_1) = f - (k - p_2 + 1)(p_2 - p_1) > f - (\ell - 1)(\ell - 1) \geq f - T = 0,$$

where the last inequality follows from the fact that $\ell \leq \sqrt{T}$. We conclude that if it is optimal to have a setup, there will be exactly one. Moreover, since $k \geq 2\ell$, this setup will be placed as early as possible (in order to balance setup and holding cost as much as possible in $[1, k]$), so in period $k - \ell + 1$.

Finally, let us compare the solutions with no or one setup in the interval $[k-\ell+1, k]$. The truncated cost $\tilde{C}^0(k)$ (resp. $\tilde{C}^1(k)$) of the solution with no (resp. one) setup equals

$$\tilde{C}^0(k) = f + \tfrac{1}{2}k(k-1) + \tfrac{f}{T-k}$$

and

$$\tilde{C}^1(k) = f + \tfrac{1}{2}(k-\ell)(k-\ell+1) + f + \tfrac{1}{2}\ell(\ell-1) + \tfrac{f}{T-k}.$$

and the cost difference $\Delta C(k)$ of the two expressions equals

$$\Delta C(k) = \tilde{C}^1(k) - \tilde{C}^0(k) = f - \ell(k - \ell).$$

This function is decreasing in k, and hence in the first iterations no setups will be generated. By setting $\Delta C(k) = 0$ we find that $k = \frac{T}{\ell} + \ell$ (w.l.o.g. we assume that ℓ divides T) and hence the first setup will appear in period $\frac{T}{\ell}+1$. This implies that the total holding cost in the first order interval equals $\frac{1}{2}\frac{T}{\ell}(\frac{T}{\ell}+1) \in \Omega(T^2)$ since ℓ is fixed. As explained before, the optimal total cost is $\mathcal{O}(T\sqrt{T})$ and hence we get the following proposition.

Proposition 8 *In case of a time window of fixed length $\ell \leq \sqrt{T}$, the worst-case ratio of the Relax and Fix heuristic with tight big M in the ULS formulation (in terms of demand) is unbounded.*

Note that in the analysis we have not used the fact that the time windows are non-overlapping. So the above property still holds in this case. Furthermore, if we let ℓ increase as T increases, the above analysis shows that a sufficient condition for having an infinite worst case ratio is to let $\ell = T^p$ with $p < \frac{1}{4}$, as then the total cost is $\Omega(T^{2-2p})$, which is larger than $\Omega(T\sqrt{T})$, while the optimal cost is $\mathcal{O}(T\sqrt{T})$. Figure 3 shows the value of the ratio between the heuristic and optimal cost for different values of the time window length ($\ell = \sqrt{T}$, $\sqrt[3]{T}$ and $\sqrt[4]{T}$). We observe a linear behavior of the ratio for $\ell = \sqrt[3]{T}$, and hence the optimality gap already tends to infinity for this case. Furthermore, it grows faster, clearly more than linearly, when $\ell = \sqrt[4]{T}$.

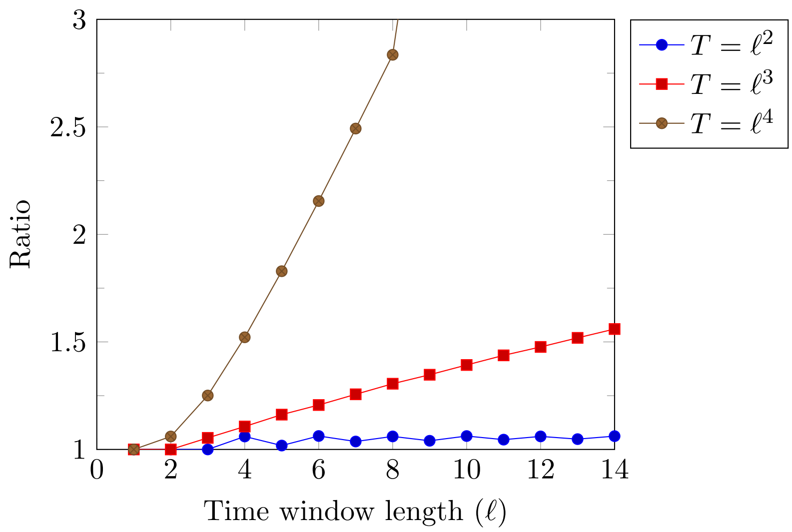

Figure 3: Optimality gap according to time window length (ℓ)

4 Observations from computational tests

In this section, we show some counterintuitive behavior of Relax and Fix heuristics, which we found when conducting some experiments. First, we study how the next production period depends on the setup cost. One would expect that the time between orders increases in the setup cost and hence that the next production period is an increasing function of the setup cost. Second, as the iterations of the Relax and Fix heuristic progress, it is expected that it becomes more attractive to have a new setup period, or in other words, the function $\Delta C(k)$ is monotonically decreasing over time. Furthermore, the longer the time window used in a Relax and Fix heuristic, the better it is expected to perform. In particular, a Relax and Fix heuristic with a time window of two periods should outperform a Relax and Fix heuristic with a time window of one period. Similarly, more overlap in the time windows is expected to have a positive impact on the quality of the solutions. Interestingly, in the next subsections we show by numerical experiments that none of the above mentioned expected behaviors may hold in general.

4.1 The next production period

In this section we show through an example that the next production period is not an increasing function in f_t. More precisely, if t (resp. t') is the next setup period for an instance with setup cost f (resp. f') with $f < f'$, then either $t < t'$ or $t > t'$ may happen. Figure 4 shows the behavior of the ratio t/T, representing the (relative) time of the second production period, according to the ratio f/T for the following instance with time-invariant parameters: $T = 50$, $c_t = c = 0, h_t = h = 1, d_t = d = 1, f_t = f \in [1, \ldots, 200]$ $\forall t = 1, \ldots, T$. We notice that the function t/T starts by increasing linearly until $f/T = 80\%$. This function remains constant until $f/T = 116\%$, then it starts decreasing gradually in a stepwise fashion. In conclusion, although one would expect an increase in the time between orders by an increase in the setup cost, this may not be the case.

4.2 Behavior of $\Delta C(k)$

We now show that the function $\Delta C(k) = C^1_{LP}(k) - C^0_{LP}(k)$ is not monotonically decreasing over time. Figure 5 depicts the evolution of $\Delta C(k)$ over time for an instance defined by

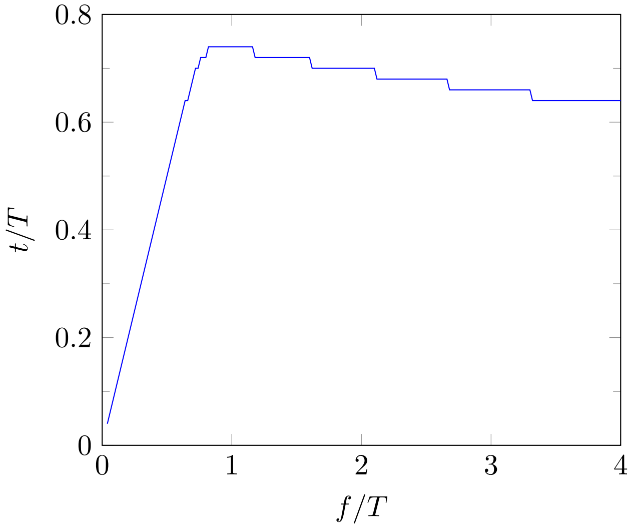

Figure 4: Timing of second setup period t/T according to setup cost f/T

$T = 50$, $c_t = 0, f_t = 50, h_t = 1, d_t = 1\, \forall t = 1, \ldots, T$. This figure shows that this difference $\Delta C(k)$ is very high at the second period $k = 2$ since we have to decide whether to produce or to store the unitary demand for one period. $\Delta C(k)$ decreases almost linearly to reach a negative value that corresponds to the next production in period 38. We can also notice that after this production period $\Delta C(k)$ starts again from a high value and forms an hyperbolic shaped function.

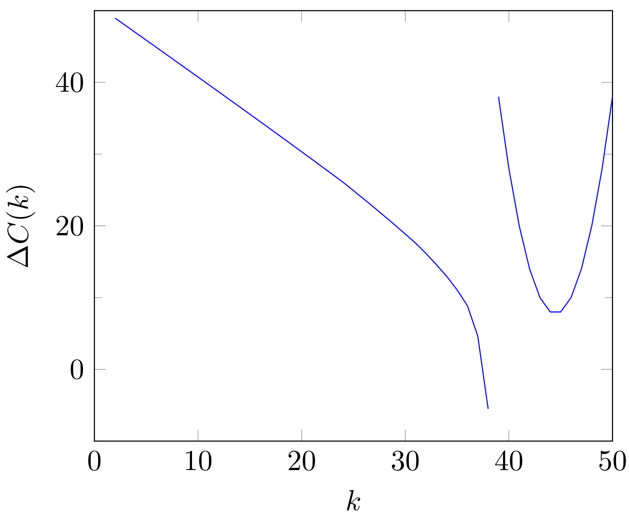

Figure 5: Evolution of $\Delta C(k)$

4.3 Impact of the time window length

We have noticed from computational tests that when all parameters are time-invariant, the Relax and Fix heuristic with a time window length equal to two is better or equivalent to the heuristic with a time window length equal to one. This observation is not always valid when increasing the length of the time window. In fact, we found a counter-example when comparing the heuristic with a time window equal to three to the one with a time window equal to two. This counter-example is defined as follows: $T = 24$, $c_t = 0, f_t = 7, h_t = 1, d_t = 1\,\forall t = 1, \ldots, T$. The cost of the heuristic with time window length equal to two is 78, while the cost with time window length equal to three is 80.

Furthermore, we have studied empirically how the optimality gaps of the heuristics with time window lengths 1 and 2 change according to the values of the setup cost. We made experiments on a large set of instances with time-invariant parameters and we obtained always the same behavior. Figure 6 shows the optimality gap between a heuristic with a time window length equal to 2 and a heuristic with a time window length equal to 1 according to the ratio f/T. We notice that when the ratio f/T is very small (lower or equal to 8%), the gap between the two heuristics is very small or equal to zero. This gap grows quickly to reach 30% when the ratio f/T is between 40% and 100%. This gap starts decreasing when f/T is higher than 100%.

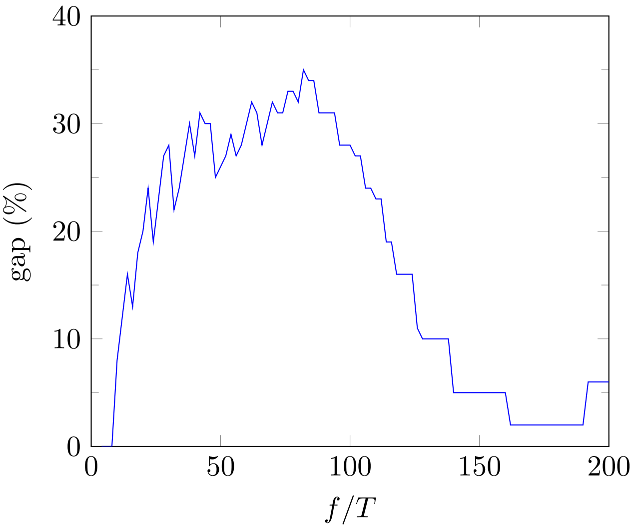

Figure 6: Optimality gap between TW=2 and TW=1 according to f/T

4.4 Overlapping time windows

In this section we study the impact of overlapping time windows (i.e., $I_{k-1} \cap I_k \neq \emptyset$) on the quality of solutions. We say that there is an overlap of m with $m > 0$ if $|I_{k-1} \cap I_k| = m$. Through a counterexample we show that more overlap does not necessarily lead to better solutions. The studied instance is defined by $T = 50$, $c_t = 0$, $f_t = 50$, $h_t = 1$, $d_t = 1$ for $t = 1, \ldots, T$. Figure 7 depicts the optimality gap of three settings of the Relax and Fix heuristic with respectively overlaps of one period ("OL=1"), two periods ("OL=2") and three periods ("OL=3") according to time window length I_k ("TW length"). We notice that there are time window lengths (e.g., TW length 10) where an overlap of one period outperforms other configurations, while for other time window lengths an overlap of two or three outperforms other configurations.

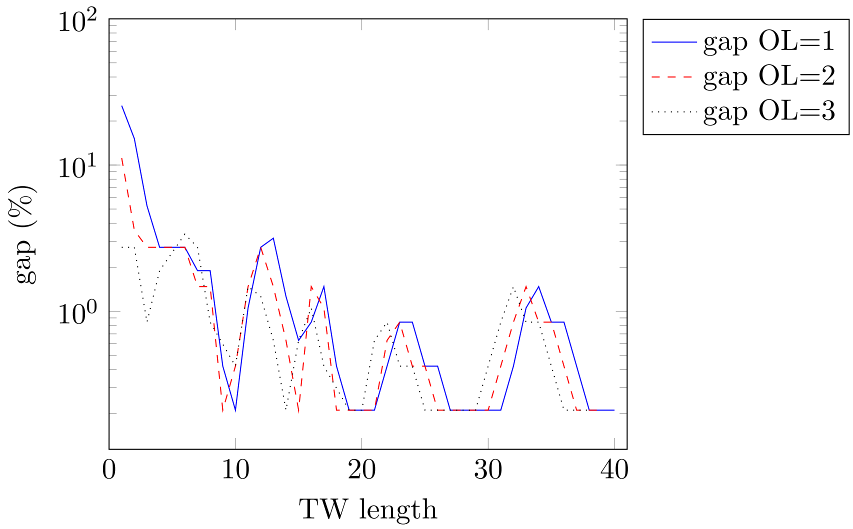

Figure 7: Overlap in time windows

4.5 Impact of capacity constraints

In this section we study the impact of capacity constraints on the quality of the solutions generated by the Relax and Fix heuristic. We introduce capacity constraints $x_t \leq C$, $\forall t \in [1, T]$, which limit the production x_t to C at each period t. Intuitively, one can expect that the tighter the capacity constraints, the better the solution quality is. The following experiments confirm this intuition. The instances are defined by $T = 50$, $c_t = 0$, $f_t = 50$, $h_t = 1$, $C \in [1, 10]$, $d_t = d \in [1, C]$ and $\ell \in [1, 10]$ for $t = 1, \ldots, T$, leading to 550 problem instances in total. Figure 8 shows the maximum optimality gap according to the ratio d/C. The maximum gap is calculated for all instances having the same ratio d/C. Figure 8 shows that the highest gaps are obtained when the ratio d/C is low ($< 30\%$) and the lowest gaps when d/C is high ($d/C > 70\%$). For average ratios of d/C ($30\% < d/C < 70\%$) gaps are in between, but interestingly we obtain a zero gap for instances with a ratio d/C of 50%. We can also notice that the maximum gap can reach almost 15%, which can be considered as relatively high. In conclusion, our analysis on a simple version of the capacitated single-item lot-sizing problems shows that when capacity constraints become tighter, the Relax and Fix heuristic performs better in general.

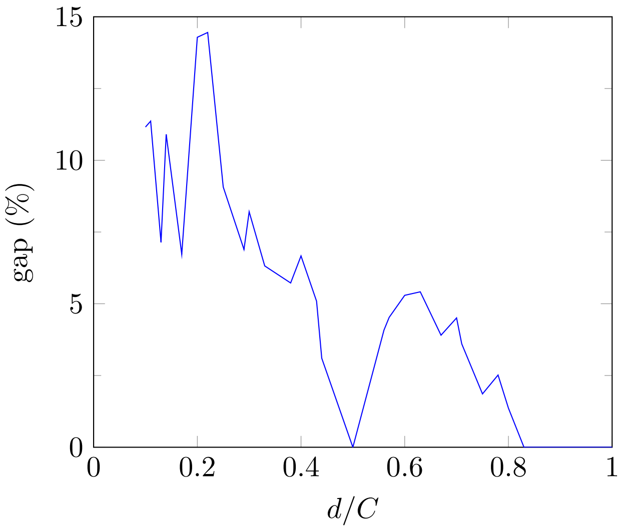

Figure 8: Impact of capacity constraints

5 Conclusion and future research

We have analyzed the worst case behavior of Relax and Fix heuristics for lot-sizing problems. We showed that even for simple instances with time-invariant parameters, the worst case ratio is not bounded for uncapacitated single-item lot-sizing problems, when using the aggregated formulation. In particular, for such problem instances with T periods, choosing a time window length smaller than $\sqrt{T}$ does not lead to a finite worst case ratio. Note that this result holds for any generalization of the uncapacitated single-item lot-sizing problem such as problems with capacities, multiple items, setup carry-over, etc. Furthermore, computational experiments have shown some counterintuitive behavior of Relax and Fix heuristics: larger time window lengths or more overlap does not necessarily lead to better performance.

Despite its worst case ratio which is not bounded for uncapacitated single-item lot-sizing problems, the use of Relax and Fix heuristics (or MIP-based heuristics in general) can be effective for large scale multi-item/multi-level capacitated lot-sizing problems. In fact, because solving the resulting monolithic models is generally time consuming, using decomposition approaches seems to be a good option to obtain good feasible solutions. The reader is referred to the references in Table 1 for more details about the performance of MIP-based heuristics for different variants of lot-sizing problems. Note that Relax and Fix heuristics do not always guarantee a feasible solution for certain lot-sizing problems. For example, this is the case for multi-item capacitated lot-sizing problems with setup times, for which the feasibility problem is already NP-hard.

Note that the performance of Relax and Fix heuristics depends on the particular mathematical formulation chosen. For further research it may be interesting to analyze the worst-case behavior of other formulations, as well as other types of problems. For example, it will be interesting to study the performance of Relax and Fix heuristics on aggregate formulations strengthened by some (l, S) valid inequalities. Interestingly, when using the facility location formulation for the uncapacitated single-item lot-sizing problem, any Relax and Fix heuristic provides an optimal solution, as the LP relaxation provides an optimal integer solution.

Finally, the performance of a Relax and Fix heuristic highly depends on the chosen settings, namely, the length of time windows and their amount of overlap. For future research, it will be interesting to develop a meta-method that aims at tuning the parameters of a Relax and Fix heuristic. This is mainly important when using this approach to solve complex industrial problems. Generally tuning these parameters is performed by trial-and-error.

References

Absi, N. and S. Kedad-Sidhoum (2007). Mip-based heuristics for multi-item capacitated lot-sizing problem with setup times and shortage costs. *RAIRO-Operations Research 41*(2), 171–192.

Akartunalı, K. and A. J. Miller (2009). A heuristic approach for big bucket multi-level production planning problems. *European Journal of Operational Research 193*(2), 396–411.

Alfieri, A., P. Brandimarte, and S. D'orazio (2002). Lp-based heuristics for the capacitated lot-sizing problem: the interaction of model formulation and solution algorithm. *International Journal of Production Research 40*(2), 441–458.

Aouam, T., K. Geryl, K. Kumar, and N. Brahimi (2018). Production planning with order acceptance and demand uncertainty. *Computers & Operations Research 91*, 145–159.

Axsäter, S. (1982). Worst case performance for lot sizing heuristics. *European Journal of Operational Research 9*(4), 339–343.

Axsäter, S. (1985). Noteperformance bounds for lot sizing heuristics. *Management Science 31*(5), 634–640.

Baker, K. R. (1989). Lot-sizing procedures and a standard data set: A reconciliation of the literature. *Journal of Manufacturing and Operations Management 2*, 199–221.

Barany, I., T. J. Van Roy, and L. A. Wolsey (1984). Strong formulations for multi-item capacitated lot sizing. *Management Science 30*(10), 1255–1261.

Belvaux, G., N. Boissin, A. Sutter, and L. A. Wolsey (1998). Optimal placement of add/drop multiplexers static and dynamic models. *European Journal of Operational Research 108*(1), 26–35.

Belvaux, G. and L. A. Wolsey (2000). bcprod: A specialized branch-and-cut system for lot-sizing problems. *Management Science 46*(5), 724–738.

Bitran, G. R., T. L. Magnanti, and H. H. Yanasse (1984). Approximation methods for the uncapacitated dynamic lot size problem. *Management Science 30*(9), 1121–1140.

Chen, H. (2015). Fix-and-optimize and variable neighborhood search approaches for multi-level capacitated lot sizing problems. *Omega 56*, 25–36.

Clark, A. R. (2003). Optimization approximations for capacity constrained material requirements planning. *International Journal of Production Economics 84*(2), 115–131.

Clark, A. R. and S. J. Clark (2000). Rolling-horizon lot-sizing when set-up times are sequence-dependent. *International Journal of Production Research 38*(10), 2287–2307.

de Araujo, S. A., M. N. Arenales, and A. R. Clark (2007). Joint rolling-horizon scheduling of materials processing and lot-sizing with sequence-dependent setups. *Journal of Heuristics 13*(4), 337–358.

dos Santos Diz, G. S., S. Hamacher, and F. Oliveira (2018). A robust optimization model for the maritime inventory routing problem. *Flexible Services and Manufacturing Journal*, 1–27.

Eppen, G. D. and R. K. Martin (1987). Solving multi-item capacitated lot-sizing problems using variable redefinition. *Operations Research 35*(6), 832–848.

Federgruen, A., J. Meissner, and M. Tzur (2007). Progressive interval heuristics for multi-item capacitated lot-sizing problems. *Operations Research 55*(3), 490–502.

Freeland, J. R. and J. Colley (1982). A simple heuristic method for lot sizing in a time-phased reorder system. *Production and Inventory Management 23*(1), 15–21.

Goren, H. G., S. Tunali, and R. Jans (2012). A hybrid approach for the capacitated lot sizing problem with setup carryover. *International Journal of Production Research 50*(6), 1582–1597.

Helber, S. and F. Sahling (2010). A fix-and-optimize approach for the multi-level capacitated lot sizing problem. *International Journal of Production Economics 123*(2), 247–256.

James, R. J. and B. Almada-Lobo (2011). Single and parallel machine capacitated lotsizing and scheduling: New iterative mip-based neighborhood search heuristics. *Computers & Operations Research 38*(12), 1816–1825.

Junqueira, R. d. Á. R. and R. Morabito (2019). Modeling and solving a sugarcan harvest front scheduling problem. *International Journal of Production Economics 2013*, 150–160.

Krarup, J. and O. Bilde (1977). Plant location, set covering and economic lot size: An $O(mn)$-algorithm for structured problems. In L. Collatz and W. Wetterling (Eds.), *Optimierung bei Graphentheoretischen and Ganzzahligen Problemen*, pp. 155–180. Birkhänser Verlag.

Kroer, L. R., K. Foverskov, C. Vilhelmsen, A. S. Hansen, and J. Larsen (2018). Planning and scheduling operating rooms for elective and emergency surgeries with uncertain duration. *Operations research for health care 19*, 107–119.

Maes, J., J. O. McClain, and L. N. Van Wassenhove (1991). Multilevel capacitated lotsizing complexity and lp-based heuristics. *European Journal of Operational Research 53*(2), 131–148.

Mercé, C. and G. Fontan (2003). Mip-based heuristics for capacitated lotsizing problems. *International Journal of Production Economics 85*(1), 97–111.

Mohammadi, M., S. Fatemi Ghomi, B. Karimi, and S. Torabi (2010). Mip-based heuristics for lotsizing in capacitated pure flow shop with sequence-dependent setups. *International Journal of Production Research 48*(10), 2957–2973.

Oliveira, B. B., M. A. Carravilla, J. F. Oliveira, and F. M. Toledo (2014). A relax-and-fix-based algorithm for the vehicle-reservation assignment problem in a car rental company. *European Journal of Operational Research 237*(2), 729–737.

Pochet, Y. and M. Van Vyve (2004). A general heuristic for production planning problems. *INFORMS Journal on Computing 16*(3), 316–327.

Ramezanian, R. and M. Saidi-Mehrabad (2013). Hybrid simulated annealing and mip-based heuristics for stochastic lot-sizing and scheduling problem in capacitated multi-stage production system. *Applied Mathematical Modelling 37*(7), 5134–5147.

Santos, M. O. and B. Almada-Lobo (2012). Integrated pulp and paper mill planning and scheduling. *Computers & Industrial Engineering 63*(1), 1–12.

Sel, Ç. and B. Bilgen (2014). Hybrid simulation and mip based heuristic algorithm for the production and distribution planning in the soft drink industry. *Journal of Manufacturing systems 33*(3), 385–399.

Stadtler, H. (2003). Multilevel lot sizing with setup times and multiple constrained resources: Internally rolling schedules with lot-sizing windows. *Operations Research 51*(3), 487–502.

Suerie, C. and H. Stadtler (2003). The capacitated lot-sizing problem with linked lot sizes. *Management Science 49*(8), 1039–1054.

Tempelmeier, H. and K. Copil (2016). Capacitated lot sizing with parallel machines, sequence-dependent setups, and a common setup operator. *OR spectrum 38*(4), 819–847.

Toledo, C. F. M., M. da Silva Arantes, M. Y. B. Hossomi, P. M. França, and K. Akartunalı (2015). A relax-and-fix with fix-and-optimize heuristic applied to multi-level lot-sizing problems. *Journal of heuristics 21*(5), 687–717.

Trigeiro, W. W., L. J. Thomas, and J. O. McClain (1989). Capacitated lot sizing with setup times. *Management science 35*(3), 353–366.

Vachani, R. (1992). Performance of heuristics for the uncapacitated lot size problem. *Naval Research Logistics 39*, 801–813.

Van den Heuvel, W. and A. P. Wagelmans (2010). Worst-case analysis for a general class of online lot-sizing heuristics. *Operations research 58*(1), 59–67.

Wolsey, L. A. (1998). *Integer Programming. Series in Discrete Mathematics and Optimization.* Wiley-Interscience New Jersey.

Wu, T., L. Shi, and J. Song (2012). An mip-based interval heuristic for the capacitated multi-level lot-sizing problem with setup times. *Annals of Operations Research 196*(1), 635–650.