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SPLITTING AND RELEVANCE: BROADENING THE SCOPE OF PARIKH'S CONCEPTS*

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ABSTRACT

When our current beliefs face a certain problem – e.g. when we receive new information contradicting them –, then we should not remove beliefs that are not related to this problem. This principle is known as "minimal mutilation" or "conservativity" [21]. To make it formally precise, Rohit Parikh [32] defined a *Relevance* axiom for (classical) theory revision, which is based on the notion of a *language splitting*.

I show that both concepts can and should be applied in a much broader context than mere revision of theories in the traditional sense. First, I generalize their application to belief change in general, and strengthen the axiom of relevance in order to make it fully syntax-independent. This is done by making use of the least letter-set representation of a set of formulas [27]. Second, I show that the logic underlying both concepts need not be classical logic and establish weak sufficient conditions for both the finest splitting theorem from [25] and the least letter-set theorem from [27]. Both generalizations are illustrated by means of the paraconsistent logic **CLuNs** and compared to ideas from [14, 36, 24].

1. Introduction

Since the publication of [32], Parikh's definition of a *language splitting* and the related axiom of *Relevance* have received quite some attention in the literature on belief revision – see e.g. [25, 28, 40, 48, 42, 43]. Although this axiom has not yet the same status as the AGM postulates for belief revision, many authors find it useful to prove that the revision operations they define obey this additional axiom – see e.g. [8, 33, 49, 43].

The main motivation for the *Relevance* axiom is that, when incorporating new information, we should remove as few beliefs as possible; hence we should certainly not remove any beliefs that have nothing to do with this new information. This principle is widely known as that of "minimal mutilation" or "conservativity" – see [21, p. 12]. The *Relevance* axiom turns this principle into a formally precise requirement.

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Let me briefly explain how it works, referring the reader to Section 3 for the exact definitions. Suppose you initially believe each of $p \wedge \sim q$, $r \vee s$, $q \vee t$. Then your beliefs can be represented equivalently (modulo classical logic) by the union of the sets $\{p\}$, $\{\sim q\}$, $\{r \vee s\}$, $\{t\}$. Accordingly, we say that the beliefs *split* the language into mutually disjoint letter sets: $\{p\}$, $\{q\}$, $\{r, s\}$, $\{t\}$. Now if you learn that $\sim p \wedge \sim r$ is the case, then the *Relevance* axiom states that this should not alter your (implicit) belief in $\sim q$ or t. So although you have to abandon the belief in p, you will stick to $\sim q$ and t.

Both the concept of language splitting and relevance are originally based on the traditional AGM model of belief revision, and hence also on classical logic as the underlying logic of our beliefs. However, as I will argue in this paper, they can and should be applied to various other types of belief change and to the dynamics of non-classical theories. This way we can combine Parikh's original idea with a pragmatic, less idealized account of belief change. As a result, we obtain a formally precise explication of the principle of minimal mutilation, which can be used to compare various operations of belief change.

Outline of this paper. I recapitulate the original definition of the finest splitting and the axiom of relevance in Section 3. After this preparatory work, I discuss how we can broaden their range of application in two ways:

(i) by applying them to other types of belief change (Section 4)
(ii) by considering beliefs whose underlying logic is non-classical (Section 5)

In both cases, I argue that the resulting generalizations yield promising and very flexible syntactic characterizations of relevant belief change. As shown in Section 6, a few basic properties suffice in order to generalize the finest splitting theorem and the least letter-set theorem to non-classical logics – both properties are crucial in order to obtain a suitable axiom of relevance based on such logics.

In Section 7, the paraconsistent logic **CLuNs** is used to illustrate the main ideas of this paper at the object-level. Next, these ideas are compared to related work (Section 8). The paper ends with conclusions and prospects for future research (Section 9).

2. Preliminaries

As announced, this paper generalizes some theorems that hold for propositional classical logic (henceforth **CL**) to a large class of logics. So let us start with the former. The language of **CL** is built up from the set of elementary letters $\mathcal{E} = \{p, q, r, \ldots, p_1, \ldots\}$, the constant $\perp$, and the connectives

$\sim, \wedge, \vee, \supset, \equiv$. Using the regular formation rules of classical logic, we obtain the set of formulas $\mathcal{W}_{\mathbf{CL}}$.

Occasionally, I shall use **CL** to denote a more general consequence relation, obtained by applying all **CL**-rules to any set of symbols in a specified formal language. With this broader notion of classical consequence, one can write such things as $\Box A \vee B, \neg B \vdash_{\mathbf{CL}} \Box A$ and $\Box A \wedge B \vdash_{\mathbf{CL}} B$. Under this reading, any formula whose outermost connective or operator is non-classical, is treated as primitive (e.g. $\Box A$ is primitive for all A).

In the remainder, I use **L** as a metavariable for any logic, i.e. a function that maps any set of formulas in a given language $\mathcal{L}_{\mathbf{L}}$ to a consequence set. Let $\mathcal{W}_{\mathbf{L}}$ be the set of closed formulas associated with **L**. Where **L** is given, I use $A, B, C, \ldots$ as metavariables for formulas in $\mathcal{W}_{\mathbf{L}}$, $\Gamma, \Delta, \Theta, \ldots$ as metavariables for sets of formulas and $\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$ as metavariables for sets of sets of formulas. $E(A)$, $E(\Delta)$ are used to denote the set of elementary letters that occur in A, resp. Δ.

Where the consequence relation $\vdash_{\mathbf{L}}$ is given and $\Gamma \subseteq \mathcal{W}_{\mathbf{L}}$, let $Cn_{\mathbf{L}}(\Gamma) = \{A \mid \Gamma \vdash_{\mathbf{L}} A\}$. Slightly abusing notation, I write $\Gamma \vdash_{\mathbf{L}} \Delta$ whenever $\Gamma \vdash_{\mathbf{L}} A$ for every $A \in \Delta$. I write $\Gamma \dashv\vdash_{\mathbf{L}} \Delta$ as an abbreviation for ($\Gamma \vdash_{\mathbf{L}} \Delta$ and $\Delta \vdash_{\mathbf{L}} \Gamma$), and $A \vdash_{\mathbf{L}} B$ for $\{A\} \vdash_{\mathbf{L}} B$.

As customary, a distinct set of metavariables $\Upsilon, \Upsilon', \ldots$ is used to refer to sets of beliefs. In this notation, Υ may be closed under a logic **L** or not.

3. Splitting and Relevance: the Classical Version

In this section, I recapitulate the definitions of language splitting and relevance from [32], and recall the finest splitting theorem from [25]. This section contains no new results but sets the stage for the remainder of this paper.

Intro. Belief revision became a subject of intensive research since the middle of the 1980s. I refer to [23] for a more gentle introduction, and will only mention some of the basic concepts here. The most common starting point for the logic of belief revision is the following question: given a set of initial beliefs Υ, and some piece of new information A that possibly contradicts Υ, how are we to revise Υ such that A can be incorporated? This is typically done by defining a revision operation $\oplus$, which is a function that maps every couple $\langle \Upsilon, A \rangle$ to a set of formulas $\Upsilon \oplus A$, called the revision set of Υ by A. We will consider other types of belief change from the next section onwards; however, as Parikh confines his discussion to revision, so will we in this section.

One may understand the logic of belief revision as a two-sided endeavour: at the "syntactic" level, one formulates postulates (also called axioms) that any operation of belief change should obey, whereas at the "semantic" level, one gives generic definitions of revision operations. An example of a postulate is the *Success* postulate for revision, which requires that for all sets of beliefs Υ and all formulas A, $A \in \Upsilon \oplus A$. An example of a "generically defined" revision operation is that of partial meet contraction – see [1]. One of the main formal challenges for the logic of belief revision is to prove representation theorems, which link these two characterizations of revision operations to each other. In addition, several scholars study the relations between various ways to define revision operations – e.g. revision operations based on entrenchment levels [16], those based on kernel contraction [20], partial meet revisions [1], model-based revision operations [18], etc.

The current paper focuses on the syntactic side of belief change. In particular, it discusses a specific rationality postulate for (classical) belief revision, and asks how we may generalize it to other types of belief change. It would be interesting to consider the question of how one may append existing semantic constructions, in such a way that this postulate can be obeyed together with a list of other standard desiderata – see [25] where this is done for classical belief revision and contraction – or whether one may even obtain representation theorems in this context. Such work requires that we consider the specific types of belief change one by one, as one can obviously not construct a single semantics for all of them.

Splitting and Relevance. In [1], eight rationality postulates for the revision of **CL**-theories, i.e. sets $\Upsilon = Cn_{\mathbf{CL}}(\Upsilon)$ are presented. As Parikh remarks in [32], these postulates are still too weak, in that they allow for the "trivial update" (henceforth $\oplus_{\mathbf{T}}$). This operation is defined as follows: if $\Upsilon \vdash_{\mathbf{CL}} \neg A$, then $\Upsilon \oplus_{\mathbf{T}} A = Cn_{\mathbf{CL}}(A)$; otherwise, $\Upsilon \oplus_{\mathbf{T}} A = Cn_{\mathbf{CL}}(\Upsilon \cup \{A\})$. As Parikh notes, 'this is unsatisfactory, because we would like to keep as much of the old information as possible [even when A contradicts the new information]. Hence the above list [= the list of postulates] needs to be supplemented to rule out the trivial update' [32, p. 3].

Parikh's positive contribution consists in the formulation of an additional postulate, i.e. the axiom of *Relevance*. The basic idea behind it is that a theory can be cut up into smaller parts which can be conceived as independent from one another [32, pp. 3-4]:

> The existing set of beliefs T may contain information about various matters. E.g. my current state of beliefs contains beliefs about the location of my children, the state of health of my teeth, and beliefs about the forthcoming election in India. In case one of my beliefs about the location of my children turns out to be false, it surely ought not to affect my beliefs about the election, since the subject matters of the two beliefs do not interact in any way.

More generally, when we are forced to change one part of a theory, this does not imply that we should do anything with the rest of the theory; we may just as well leave it unaltered. If we receive information that relates to only some parts of our theory, then the relevance axiom stipulates that we should leave the rest of the theory as it is.

To turn this idea into a formally precise concept, Parikh defines the so-called *splitting* of a theory Υ. This name may be somewhat misleading, as a splitting is in fact a partition[1] of the *letter set* $\mathcal{E}$ rather than of Υ; however, $\mathcal{E}$ is split in a way that is relative to the specific theory under consideration.

Definition 1 ([28]: Def. 3.1). *Let $\mathbb{E} = \{\Lambda_i\}_{i\in I}$ be a partition of $\mathcal{E}$. We say that $\mathbb{E}$ is a* splitting *of Γ iff there is a $\Delta = \bigcup_{i\in I} \Delta_i$ such that each $E(\Delta_i) \subseteq \Lambda_i$ and $\Delta \dashv\vdash_{\mathbf{CL}} \Gamma$.*[2]

Example 1. *Let $\Upsilon = \{(p \vee q) \wedge r, \sim r \vee s, q \vee t, r \vee u\}$. Note that this set is* **CL***-equivalent to each of the following sets:*

$$\Upsilon' = \{p \vee q, q \vee t, r, \sim r \vee s\}$$
$$\Upsilon'' = \{p \vee q, q \vee t, r, s\}$$

From Υ, Υ^{I} and Υ^{II} respectively, we may generate the following partitions of $\mathcal{E}$:

$$\mathbb{E}1(\Upsilon) = \{\{p, q, r, s, t, u\}\} \cup \{\{A\} \mid A \in \mathcal{E} - \{p, q, r, s, t, u\}\}$$
$$\mathbb{E}2(\Upsilon) = \{\{p, q, t\}, \{r, s\}\} \cup \{\{A\} \mid A \in \mathcal{E} - \{p, q, t, r, s\}\}$$
$$\mathbb{E}3(\Upsilon) = \{\{p, q, t\}, \{r\}, \{s\}\} \cup \{\{A\} \mid A \in \mathcal{E} - \{p, q, t, r, s\}\}$$

Consequently, each of these are splittings of Υ.

$\mathbb{E}$ is *at least as fine as* $\mathbb{E}'$ iff every cell of $\mathbb{E}'$ is the union of cells of $\mathbb{E}$; $\mathbb{E}$ is finer than $\mathbb{E}$ iff it $\mathbb{E}$ is at least as fine as $\mathbb{E}'$ but the converse fails. Note that if $\mathbb{E}$ is a splitting of Γ, and $\mathbb{E}$ is finer than the partition $\mathbb{E}'$ of $\mathcal{E}$, it immediately follows that $\mathbb{E}'$ is also a splitting of Γ (see [32, pp. 4-5]). We say that $\mathbb{E}$ is a *finest splitting* of Γ iff there is no splitting $\mathbb{E}'$ of Γ that is finer than $\mathbb{E}$.

Example 2. *Take Υ from Example 1. Note that $\mathbb{E}_2(\Upsilon)$ is finer then $\mathbb{E}_1(\Upsilon)$, and $\mathbb{E}_3(\Upsilon)$ is finer then $\mathbb{E}_2(\Upsilon)$. Provably, $\mathbb{E}_3(\Upsilon)$ is a finest splitting of Υ.*

Parikh shows in [32] that every finite set $\Gamma \subseteq \mathcal{W}_{\mathbf{CL}}$ has a unique finest splitting. Kourousias and Makinson generalized this result to the infinite case, thus obtaining the following crucial theorem:

[1] A *partition* $\mathbb{A} = \{\Lambda_i\}_{i\in I}$ of a set Θ is a set of non-empty, pairwise disjoint sets such that $\bigcup_{i\in I}\{\Lambda_i\} = \Theta$. In this notation, the sets Λ_i are called the *cells* of $\mathbb{A}$.

[2] The idea of a splitting originates in [32]. I use Makinson's definition because it includes the case where Γ is infinite.

Theorem 1 ([25]: Th. 2.4). *Every* $\Gamma \subseteq \mathcal{W}_{\mathbf{CL}}$ *has a unique finest splitting.*

On the basis of this fact, we may use the notion of a finest splitting to define relevance in the context of belief revision:

Definition 2. *Let* $\mathbb{E}$ *be the finest splitting of* Υ. *We say that a formula* B *is irrelevant to* A *modulo* Υ *iff for every cell* $\Lambda_i \in \mathbb{E}$: $\Lambda_i \cap E(A) = \emptyset$ *or* $\Lambda_i \cap E(B) = \emptyset$.

Parikh himself only considers the case where Υ is a theory, i.e. $\Upsilon = Cn_{\mathbf{CL}}(\Upsilon)$ – I return to this point in the next section. His axiom of relevance reads as follows:

$\mathbf{P}_{\mathrm{or}}$ *Original Relevance:* If $B \in \Upsilon$ is irrelevant to A modulo Υ, then $B \in \Upsilon \oplus A$.

As shown in [32], postulate $\mathbf{P}_{\mathrm{or}}$ is compatible with the six basic AGM postulates for theory revision from [1], whenever Υ is consistent.[3] The basic idea behind the proof of this fact is that one defines a revision operator $\oplus_{\mathbf{P}}$ from an AGM-obedient revision operator $\oplus$ as follows:

$$\Upsilon \oplus_{\mathbf{P}} A = Cn_{\mathbf{CL}}(\Upsilon_1 \cup (\Upsilon_2 \oplus A))$$

where Υ_1 consists of all beliefs in Υ that are not relevant to A modulo Υ, and Υ_2 consists of all beliefs in Υ that are relevant to A modulo Υ. We refer to [32] for the exact details. In [43], eight distinct revision operations are defined, each of which satisfy all the six AGM postulates and $\mathbf{P}_{\mathrm{or}}$.

We finish this section with an example that illustrates the power of this relevance axiom.

Example 3. *Consider the revision of* $Cn_{\mathbf{CL}}(\Upsilon)$ *by* $\sim r$. *If* $\boldsymbol{P}_{\mathrm{or}}$ *is obeyed, then* $p \vee q$, $p \vee t$ *and* s *are in the revision set, since all of them are in* $Cn_{\mathbf{CL}}(\Upsilon)$, *and none of them are relevant to* $\sim r$ *modulo* Υ. *However,* $r \vee u$ *may not be in the revision set, as this formula shares letters with* r *itself.*

Consider now the revision of $Cn_{\mathbf{CL}}(\Upsilon)$ *by* $\neg p \wedge \neg q$. *In this case, both* $p \vee q$ *and* $p \vee t$ *are relevant to* $\neg p \wedge \neg q$ *modulo* Υ, *whence they are not required to be in the revision set. On the other hand, the belief* r *is not relevant to* $\neg p \wedge \neg q$ *modulo* Υ, *and should be upheld in order to satisfy* $\boldsymbol{P}_{\mathrm{or}}$.

4. Generalization 1: Belief Change

In this section, I will show that the concepts of splitting and relevance have interesting applications in various kinds of belief change other than revision

[3] In case Υ is consistent, $\mathbf{P}_{\mathrm{or}}$ clashes with the *Consistency* requirement; we return to this point in Section 5.2.

of theories in the traditional sense, generalizing some of the definitions from the preceding section accordingly. In addition, it is shown that one may strengthen the *Relevance* axiom and thereby make it fully syntax-independent in each of its three arguments, making use of Makinson's least letter-set theorem [28, 27].

4.1. *Belief Bases*

It is well-known that the research on belief revision is divided into two competing approaches: one in terms of theories (sets that are closed under classical logic), and the other in terms of belief bases (arbitrary sets of formulas). The latter is obviously more general, and several authors have argued why it is more suitable for certain purposes. So a first question that arises is whether we can also apply the relevance axiom to belief base revision. I will not be able to answer this question in full here, but point at some interesting problems that arise, which should be tackled in future work.

Kourousias and Makinson [25] explain that the problem of trivial updates is easily transposable to belief bases, i.e. (in their framework) sets Υ that are not closed under **CL**. For example, when revising the base $\Upsilon_1 = \{p \wedge q\}$ by $\sim p$, there are "rational" (in the sense of [21, Chapter 3, Section 1]) belief operations that yield $\sim p$ as the only resulting belief, hence removing both the implicit beliefs p and q.

However, Parikh only intends to apply this axiom to **CL**-theories – for bases, it does not solve the above problem. Let $\Upsilon = \{p \wedge q\}$. Then the formula $p \wedge q \in \Upsilon$ *is* relevant to $\sim p$ modulo Υ. Hence if we take $\mathbf{P}_{\mathrm{or}}$ literally, there is no problem in dropping $p \wedge q$, which is the only belief in Υ. Nevertheless, Kourousias and Makinson continue to work with the original version of Parikh's relevance axiom, even though they apply it to belief bases in general.

In order to deal with belief bases in a more appropriate way, we may generalize the axiom as follows:[4]

$\mathbf{P}_{\mathrm{b}}$ *Relevance for Bases:* If $B \in Cn_{\mathbf{CL}}(\Upsilon)$ is irrelevant to A modulo Υ, then $B \in Cn_{\mathbf{CL}}(\Upsilon \oplus \Psi)$.

Before we consider it in more general terms, let us show how $\mathbf{P_b}$ works by means of a simple example.

Example 4. *Consider the revision of* $\Upsilon = \{(p \vee q) \wedge r, \sim r \vee s, q \vee t, r \vee u\}$ *by* $\sim r$*. If* $\mathbf{P_b}$ *is obeyed, then this implies that* $p \vee q$, $p \vee t$ *and* s *are* **CL**-*derivable from the revision set.*

[4] Whenever Υ and $\Upsilon \oplus A$ are closed under **CL**, as in the traditional AGM-approach, this formulation reduces to Parikh's original axiom.

Note that if Υ and Υ' are **CL**-equivalent belief bases, then they have the same finest splitting, whence relevance modulo a revision of Υ by A is equivalent to relevance modulo a revision of Υ' by A. Hence relevance is independent of the way we represent Υ. It is not independent of the way we represent A – I return to this point in Section 4.3.

However, $\mathbf{P}_\mathrm{b}$ does not imply that the revision of distinct, yet equivalent belief bases will always bring us into one and the same end state. It poses a lower bound on the revised base, but depending on the initial belief base, one may prefer specific ways to satisfy this lower bound. In Hansson's terms, this means that the "dynamic difference" between two (classically equivalent) belief bases is not contradicted by $\mathbf{P}_\mathrm{b}$.

For instance, compare the belief base Υ from Example 4 to $\Upsilon' = \{(p \vee q) \wedge r, {\sim}r \vee s, q \vee t\}$. If we revise Υ by ${\sim}r$, then we may have good reasons to uphold the belief in $r \vee u$ (e.g. because we had independent reasons for this belief), and hence we may have u in the revised belief base. On the other hand, if we revise Υ' by ${\sim}r$, then we are in a very different situation (even though Υ and Υ' are classically equivalent), and we have no reasons to put u in the revised belief base.

Another point to note is that $\mathbf{P}_\mathrm{b}$ is not in general compatible with the postulates of *Inclusion* (**I**), *Success* (**S**) and *Consistency* (**C**) for belief base revision from Hansson's [21]. These can be spelled out as follows:

I $\quad \Upsilon \oplus A \subseteq \Upsilon \cup \{A\}$

S $\quad A \in \Upsilon \oplus A$

C $\quad$ if A is consistent, then $\Upsilon \oplus A$ is consistent

For instance, suppose that $\Upsilon = \{p \wedge q\}$ and $A = {\sim}p$. Let $\oplus$ be a revision operation on belief bases that satisfies each of **I**, **S** and **C**. By **I** and **S**, $\{{\sim}p\} \subseteq \Upsilon \oplus A \subseteq \{p \wedge q, {\sim}p\}$. So by **C**, $\Upsilon \oplus A = \{{\sim}p\}$. But this means that $\oplus$ does not validate $\mathbf{P}_\mathrm{b}$, since q is lost, although it is not relevant to A modulo Υ.

One way to combine (the spirit of) the axioms for belief base revision and that of *Relevance*, is by reformulating *Inclusion* as follows:

I′ $\quad Cn_{\mathbf{CL}}(\Upsilon \oplus A) \subseteq Cn_{\mathbf{CL}}(\Upsilon \cup \{A\})$

With this axiom, we still ensure that the revised belief state can never contain more than what is contained in the combination of the old beliefs and the new information. It remains an open problem whether this change suffices in order to restore compatibility with all of Hansson's postulates for base revision with $\mathbf{P}_\mathrm{b}$.

In any case, the incompatibility of $\mathbf{P}_\mathrm{b}$ with existing postulates for belief base revision should not be seen as a sufficient reason to reject the former in favour of the latter. Indeed, a relevant revision may require us to analyse and "reformulate" some of our initial beliefs. But this does not imply that the central unit of change is really a theory, or that **P** only makes sense in the context of theory revision. Consider the paper you are reading at the moment, supposing that it represents my current beliefs. Obviously, it consists of a finite stock of statements and not every logical consequence of it is (or can be) made explicit. If I were to revise this paper upon learning new information, the result would still be a finite entity. Still, it makes perfect sense for me to revise the paper in such a way that I rescue as much of it as possible – even if this means that I have to reformulate some of my earlier statements, to analyse certain previous claims in order to keep parts of them, etc.

This observation opens up the space for a new question: how can we revise a given belief base Υ in such a way that (i) we obey *Relevance* (as specified by $\mathbf{P}_b$), and (ii) we perform a minimal analysis on Υ? This is a difficult problem, and it seems to lead us far beyond the constructions in terms of maximal consistent subsets that are common in the logic of belief revision. It seems particularly important for the implementation of the *Relevance* axiom in computationally feasible models of belief revision.

4.2. *Beyond Revision*

A second generalization concerns the notion of *revision* itself. In the belief revision literature, there are three classical operations of belief change: revision, contraction and expansion. Roughly speaking, contraction is the (mere) removal of beliefs from Υ; expansion is (merely) adding beliefs to Υ. Revision is usually reduced to a combination of contraction and expansion – see e.g. [21] for the many details.

Obviously, for the case of expansion, the *Relevance* axiom makes little sense, as none of the initial beliefs are in danger of being removed. As for contraction, it should be noted that Kourousias and Makinson [25] already apply the relevance axiom to this operation, showing that any operation of revision that is based on a relevant operation of contraction warrants relevance of the former.

However, apart from the three standard operations, there is a whole range of other operations as well [22]: semi-revision (which is itself a generalization of screened revision from [26]), impure contraction, (various sorts of) consolidation, abduction, …. In each of these cases, the main idea is still that we work on the basis of a set of (propositional) formulas Υ, which we have to change in view of a given formula A – the latter may either represent new information, a formula which we want to contract, a phenomenon which we want to explain, etc. So we may represent belief change operations by $\Upsilon \bullet A$, where the reading of $\bullet$ is disambiguated by the context.

One may also consider the contraction of or (semi-)revision by *sets* of formulas Ψ instead of just single formulas A. Such operations have been called multiple contraction and multiple revision, and were studied in detail in [15].

This brings us to the following, general notion of belief change. We have a set of initial beliefs $\Upsilon \subseteq \mathcal{W}_{\mathbf{CL}}$, and a set of formulas $\Psi \subseteq \mathcal{W}_{\mathbf{CL}}$ that urge us to change Υ. Put differently, one way or another, Ψ constitutes a problem for Υ. Finally, the set $\Upsilon \bullet \Psi$, again a subset of $\mathcal{W}_{\mathbf{CL}}$, represents the solution of this problem.

Obviously, not all the standard AGM postulates apply to belief change in general. One can distinguish between postulates that highlight particular features of a certain operation, and postulates which seem to work for any type belief change. Examples of the former are the *Succes* postulate (which obviously fails for semi-revision) and the *Closure* postulate (which makes little sense for base revision); examples of the latter are the principle of *categorical matching* mentioned above and certain requirements that concern the non-triviality and consistency of the outcome of a belief change.[5]

The *Relevance* axiom is rather of the second type, even though it is a very precise requirement. It relies essentially on the notion of language splittings in view of Υ – which has nothing to do with the kind of change we are dealing with – and the basic principle of minimal mutilation, which is itself a universal, yet rather vague principle of belief change.

It should be stressed that this last point only applies to belief change, conceived as the change of a belief state, where the subject of the beliefs is supposed to remain the same. This is different from belief *update*, where it is assumed that the beliefs concern a changing world – we refer to [23] for a discussion of this distinction. As shown in [31], Parikh's relevance axiom is not as easily applicable in the context of belief update.

4.3. *Full Syntax-Independency*

In this short section, I introduce a variant of the relevance axiom, which has to do with the specific role of $E(A)$ in the definition of relevance, given an operation of belief change. Consider the revision of $\Upsilon = \{p \wedge q\}$ by $\sim q \wedge (p \vee \sim p)$. Note that the new information is **CL**-equivalent to $\sim q$, and hence one would expect that in this situation, the relevance axiom requires that p is upheld. However, since $E(\sim q \wedge (p \vee \sim p)) = \{p, q\}$, both p and q are relevant to $\sim q \wedge (p \vee \sim p)$ modulo Υ.

[5] It is difficult to be more precise here, since the exact formulations diverge again from one operation to the other: for revision, consistency is usually required as long as the new information is consistent; for contraction, it is required whenever the contraction formula is not tautological.

As this example may seem rather simplistic, recall that we are not only considering belief change in view of single formulas, but also in view of (possibly complex, infinite) sets of formulas Ψ. One can easily imagine that such Ψ contain certain redundant letters, which we want to ignore in our concept of relevant belief change.

This problem is solved by making use of the notion of a least letter-set representation from [27, 28], which may be conveniently formulated as follows:[6]

Definition 3. *Ψ^* is a least letter-set representation of Ψ iff (i) $\Psi^* \dashv\vdash_{\mathbf{CL}} \Psi$ and (ii) for every Δ such that $\Delta \dashv\vdash_{\mathbf{CL}} \Psi$, $E(\Delta) \subseteq E(\Psi^*)$. Let $E^*(\Psi) = E(\Psi^*)$, where Ψ^* is an arbitrary least letter-set representation of Ψ.*

We call $E^*(\Psi)$ the least letter-set of Ψ. As shown in [27], every set of formulas Ψ has a least letter-set representation Ψ^*. For instance, in **CL**, a least letter-set representation of $\Theta = \{p, \neg p \vee q, q \vee r, s \supset t\}$ is $\Theta' = \{p, q, s \supset t\}$. The corresponding *least letter-set* is thus $\{p, q, s, t\}$.

If $\Gamma \vdash_{\mathbf{CL}} \bot$, then Γ has only one least letter-set representation, viz. $\{\bot\}$, and hence its least letter-set is empty. On the other hand, every contingent set of formulas Γ has infinitely many least letter-set representations but only one least letter-set. For instance, $\{p\}$, $\{p \wedge p\}$, $\{p \wedge p \wedge p\}$, ... are all least letter-set representations of $\{p\}$.

By adding the idea of a least letter-set representation to our previous definition, we make sure that only those letters in Ψ that are non-redundant, will be taken into account. This way we obtain the following axiom:

P *Relevance:* Where Ψ^* is an arbitrary least letter-set representation of Ψ: if $B \in Cn_{\mathbf{CL}}(\Upsilon)$ is irrelevant to Ψ^* modulo Υ, then $B \in Cn_{\mathbf{CL}}(\Upsilon \bullet \Psi)$.

5. Generalization 2: Non-Classical Logics

In this section, I argue for a pluralistic approach towards the underlying logic of belief change (Section 5.1). In Section 5.2 it is shown that the classical *Relevance* axiom leads to absurdities in the case of inconsistent theories. I proceed by showing what a relativized axiom of **L**-*Relevance* looks like, where **L** is an arbitrary logic (Section 5.3). Finally, I discuss the relation between language splittings that are defined in terms of different logics (Section 5.4).

[6] We refer to Section 6.4 for some more details on this definition and the related least letter-set theorem.

5.1. *Theory Dynamics and Pluralism*

There are several types of (logical) pluralism in the literature and various ways to motivate these – see [10] for a short overview. As this is not the main topic of the current paper, I will just state my own position and omit many details, referring to the literature for more elaborate discussion.[7]

In his [10], Cook distinguishes (inter alia) between two kinds of pluralism:

- *dependent* logical pluralism – whether or not a logic is suitbale, is relative to certain conditions, e.g. the type of reasoning we want to explicate, where we draw the boundary between logical and non-logical entities in the language, etc.
- *simple* logical pluralism – even if we have an exact application in mind, and if we have fixed the boundary between logical and non-logical language, there may still be different logics that adequately capture our notion of logical consequence in this context.

What I am after is merely a dependent pluralism with regards to the underlying logic of beliefs. This suffices for present needs, whence I remain neutral concerning the second kind of pluralism.

Following [39], logics can be conceived as *models* that represent certain types of reasoning. Under this interpretation, taking **L** as the underlying logic of our beliefs, implies that one assumes **L** to be representative of the way we should interpret these beliefs and reason with them. Note the normative component – it is not because models represent something, that they are confined to merely representing reasoning as a psychological phenomenon. However, to serve as an interesting model, **L** should itself of course also somehow relate to "actual reasoning", or at least to what we consider as "good examples" of it.

Given this general aim of logic, there are good reasons to be pluralistic concerning what counts as a correct logic. That is, models, and a fortiori logics, abstract from and idealize certain aspects of the phenomena which they represent. This is what makes logic interesting in the first place. However, it is not always clear where to abstract and what to idealize; this usually depends on the specific problem at hand, what we want to achieve by our formalism. As concerns abstraction, there is e.g. no unique best way to draw the boundary between logical and non-logical parts of the language (cf. [44]). As for idealization, certain presuppositions may be harmless given a certain application, but may significantly distort our picture of other types of reasoning. For instance, in one context, one may safely ignore the possibility of inconsistent beliefs, whereas in others want may consider this as an unjustified idealization.

[7] I am indebted to Neil Coleman for his helpful suggestions concerning this section.

So in general, what counts as a sufficiently suitable model depends on the specific desiderata one has in mind for that model. Consequently, several criteria will play a role when we pick a logic: it should be strong and rich enough, yet still not trivialize certain types of belief sets; it should have certain meta-properties; it should handle certain connectives in a specific way; it should be transparent and easy to work with, etc.

In other words, the kind of logical pluralism I have in mind boils down to the following claim: there just seems not to be one single logic that can do all the jobs we expect from logical models in the context of belief revision, in a satisfying and interesting way. If we want to model the way humans should reason with theories, with the tools that are currently available, then we cannot content ourselves with the study of one single logic such as **CL**.

Note that this pluralism does not say anything about the metaphysical question whether there is one "correct" logic of beliefs, or one relation we can truly call "logical consequence". It also does not prohibit that one confines oneself to **CL** for the sake of argument – this is perfectly alright, as long as one is aware that this rests on a pragmatic decision of the logician, not on some a priori valid principles.

Once we grant that the logic of belief *change* should be based on the underlying logic of our beliefs, then in view of the preceding, we should also be pluralistic with regards to the logic of belief change. Indeed, one can easily think of richer theories based on a modal logic, theories in which probabilities can be expressed, etc. In either case, **CL** seems just too poor to explicate why and how we may change our theories.

In fact, the standard AGM approach already envisions certain non-classical theories. In [1], the underlying logic is just assumed to be a compact *supraclassical* Tarski-logic **L** which satisfies "introduction of disjunction in the premises".[8] So at least in this sense, the idea of logical pluralism was embraced from the start.[9]

But there are also good reasons to consider beliefs whose logic does not validate certain classical inferences. As pointed out by several authors, inconsistent belief bases are a fact of life – see e.g. [24, 14, 36, 11, 34, 41]. Especially when large databases are constructed, it becomes very hard to avoid inconsistencies altogether. Likewise, it is commonly acknowledged that even our most reliable scientific theories can turn out to be inconsistent. As **CL** trivializes such theories, one has to work with a weaker underlying logic to make meaningful use of these theories, at least until one has been

[8] This property reads as follows: if $B \in Cn_{\mathbf{L}}(\Gamma) \cup \{A_1\})$ and $B \in Cn_{\mathbf{L}}(\Gamma \cup \{A_2\})$, then $B \in Cn_{\mathbf{L}}(\Gamma \cup \{A_1 \vee A_2\})$.

[9] Renata Wasserman makes the same point in [46], which provides a tentative overview of non-classical approaches to AGM belief revision.

able to replace them with new, consistent ones. Consequently, several scholars have tried to transfer formal results on belief revision to the paraconsistent setting – see the references at the start of this paragraph.

Some readers may wonder what paraconsistent belief revision would look like. That is, if contradictions do not result in triviality, then the difficulty of incorporating new, possibly conflicting information seems to disappear. I refer to Priest's [34] for a lengthy discussion of this objection. The bottom line of his argument is that, even though adding new, conflicting information does not lead to triviality in the context of paraconsistent logic, one may still have good (epistemological) motivations to avoid making one's theory (more) inconsistent, and hence to withdraw certain beliefs when incorporating new ones. In fact, going paraconsistent seems to be the only way one can model the fact that sometimes, we remove one contradiction from a theory, whereas we leave other contradictions unaltered – I return to this point in Section 7.

Let us now return to the axiom of *Relevance*. As I will show in Section 5.3, this axiom can work perfectly well without the presupposition that our theories are based on **CL**. It only depends on the intuition of minimal change and the concept of a language splitting. As will become clear, the latter can be easily relativized to any underlying logic **L**.

It should be noted that such a relativization is not just straightforward, but also *necessary*, if we want to apply Parikh's concept of relevance to non-classical theories. For theories based on a paraconsistent logic, classical relevance is too strong – this will be explained in detail below. For theories based on extensions of **CL**, the classical relevance axiom is too weak, since **CL** is not able to analyze formulas whose outermost connective is non-classical.

5.2. *Relevance and Inconsistency*

In this section I briefly explain why the classical relevance axiom **P** leads to absurdities, when applied to inconsistent theories or belief bases. This fact motivates the search for variations on **P**, in terms of a different (paraconsistent) underlying logic **L**.

I start with theories. Let $\mathcal{W}^l$ denote the set of literals, i.e. propositional letters and their negation. Suppose now that $\Upsilon = Cn_{\mathbf{CL}}(\Upsilon)$ and $\Upsilon \vdash_{\mathbf{CL}} \bot$. Note that $A \in \Upsilon$ for every $A \in \mathcal{W}^l$, whence $\mathcal{W}^l \dashv\vdash_{\mathbf{CL}} \Upsilon$. It follows that the finest splitting of Υ is $\mathbf{E} = \{\{A\} \mid A \in \mathcal{E}\}$. This means that relevance modulo Υ reduces to mere letter-sharing: $B \in \Upsilon$ is relevant to A modulo Υ iff $E(B) \cap E(A) \neq \emptyset$. As a result, a revision operation $\Upsilon \oplus A$ that obeys **P** would result in (a superset of) the set $\{B \in \mathcal{W}^{\perp} \mid E(B) \cap E^{*}(A) = \emptyset\}$. Hence, such a revision operation would result in something close to plain triviality.

So how about belief bases? Suppose again that $\Upsilon \vdash_{\mathbf{CL}} \bot$. By the same reasoning as in the previous paragraph, $\mathcal{W}^l \dashv\vdash_{\mathbf{CL}} \Upsilon$ and the finest splitting of Υ is $\mathbf{E} = \{\{A\} \mid A \in E\}$. If **P** is obeyed, this means that for every $B \in \mathcal{W}^l$, B has to be in the **CL**-consequence set of $\Upsilon \oplus A$ whenever $E(B) \cap E^*(A) \neq \emptyset$. Hence it is required that $\Upsilon \oplus A$ is inconsistent; but more importantly, it suffices to take *any* inconsistent set Υ', in order to obey the axiom **P**. Arguably, this requirement is far too liberal to receive the status of a rationality postulate.

This does not mean that the intuition behind Parikh's relevance axiom is not applicable to inconsistent sets of beliefs. Consider the belief base $\Upsilon = \{p \wedge q, r, \sim r\}$, and suppose we have to contract it by $q \vee r$. Even though one has to remove both q and r, one can readily argue that p is not relevant to this particular contraction. This has little to do with the fact that Υ is inconsistent.

Note that even a very weak paraconsistent logic will usually validate Simplification (from $A \wedge B$, infer A and B).[10] Hence in the above example, every such logic will allow us to derive p from $p \wedge q$, and hence consider p as separable from q in Υ. More generally, a logic can be (fully) paraconsistent, yet still allow us to analyse our set of initial beliefs to some extent, and hence obey a certain degree of relevance. So we can obtain a strong, but also non-trivializing relevance axiom, if we weaken our standard of deduction in such a way that inconsistencies do not cause us to believe anything. This will be illustrated in Section 7 by means of the strong paraconsistent logic **CLuNs**.

5.3. *Generalizing the Definitions*

To obtain a non-classical concept of relevance, we first need to relativize the definition of a splitting and the least letter-set set, as follows:

Definition 4 (L-splitting). *Let $\mathbb{E} = \{\Lambda_i\}_{i\in I}$ be a partition of $\mathcal{E}$. We say that $\mathbb{E}$ is a* **L**-splitting *of Γ iff there is a $\Delta = \bigcup_{i\in I} \Delta_i$ such that each $E(\Delta_i) \subseteq \Lambda_i$ and $\Delta \dashv\vdash_{\mathbf{L}} \Gamma$.*

Definition 5. *Ψ^* is a least letter-set representation of Ψ in* **L** *iff (i) $\Psi^* \dashv\vdash_{\mathbf{L}} \Psi$ and (ii) for every Δ such that $\Delta \dashv\vdash_{\mathbf{L}} \Psi$, $E(\Psi^*) \subseteq E(\Delta)$. Let $E^*_{\mathbf{L}}(\Psi) =_{\mathrm{df}} E(\Psi^*)$, where Ψ^* is an arbitrary least letter-set representation of Ψ in* **L**.

Note that these definitions are identical to Definitions 1 and 3 respectively, in case **L** = **CL**.

Suppose now that every $\Gamma \subseteq \mathcal{W}_{\mathbf{L}}$ has a finest **L**-splitting. We may then define **L**-relevance simply by replacing **CL** with **L** in Definition 2:

[10] For some examples of logics that do *not* validate Simplification, see [3] – this paper contains an interesting classification of logics in which some connectives other than $\sim$ behave non-classically in various ways.

Definition 6 (L-relevance). *Let $\mathbb{E}$ be the finest* **L***-splitting of* Υ. *We say that a formula B is* **L***-irrelevant to* Ψ *modulo* Υ *iff for every cell* $\Lambda_i \in \mathbb{E}$*:* $\Lambda_i \cap E(\Psi) = \emptyset$ *or* $\Lambda_i \cap E(B) = \emptyset$.

The above ingredients finally allow us to define an axiom of **L**-relevance:

$\mathbf{P_L}$ Where Ψ^* is a least letter-set representation of Ψ in **L**: If $B \in Cn_{\mathbf{L}}(\Upsilon)$ is not **L**-relevant to Ψ modulo Υ, then $B \in Cn_{\mathbf{L}}(\Upsilon \bullet \Psi)$.

The crucial properties we need for these definitions to be consistent, is that every $\Upsilon \subseteq \mathcal{W}_{\mathbf{L}}$ has a finest **L**-splitting, and that each $\Psi \subseteq \mathcal{W}_{\mathbf{L}}$ has a least letterset representation. In Section 6, I establish and discuss sufficient conditions for both these properties.

5.4. *Sufficient Criteria for (Ir)relevance*

Once the notion of a splitting is generalized to a class of logics **L**, a question that immediately comes to mind is when and how **L**-splittings relate to **L**$'$-splittings, for two logics **L** and **L**$'$. As may be expected, the stronger the underlying logic, the finer it allows us to split $\mathcal{E}$ in view of Γ:[11]

Theorem 2. *If* **L** *is at least as strong as* **L**$'$, *then any* **L**$'$*-splitting of* Γ *is a* **L***-splitting of* Γ.

Proof. Suppose the antecedent holds and $\mathbb{E} = \{E_i\}_{i\in I}$ is an **L**$'$-splitting of Γ. Hence there is a $\Delta = \bigcup_{i\in I} \Delta_i$ such that $\Delta \dashv\vdash_{\mathbf{L}'} \Gamma$ and each $E(\Delta_i) \subseteq E_i$. It follows from the supposition that $\Delta \dashv\vdash_{\mathbf{L}} \Gamma$. But then $\mathbb{E}$ is a **L**-splitting of Γ. ■

Example 5. *Let* $\Upsilon = \{(p \vee q) \wedge {\sim}s, \Box(\neg q \wedge r), \Box(s \vee t), \Box\Box(s \supset t)\}$. *We consider splittings of* Υ *modulo four logics:* **CL**, *the minimal modal logic* **K** *(obtained by adding the necessitation rule and the distribution axiom to* **CL***), Feys' logic* **T** *(obtained by adding the truth axiom to* **K***), and* **S4** *(obtained by adding* $\Box A \supset \Box\Box A$ *to* **T***). I refer to* [17] *for the full axiomatic characterizations and semantics of each of these systems.*

Note that each of the following holds:[12]

(i) $\Upsilon \dashv\vdash_{\mathbf{CL}} \{p \vee q, \Box(\neg q \wedge r), {\sim}s, \Box(s \vee t), \Box\Box(s \supset t)\}$ *(by* Simplification*)*

(ii) $\Upsilon \dashv\vdash_{\mathbf{K}} \{p \vee q, \Box\neg q, \Box r, {\sim}s, \Box(s \vee t), \Box\Box(s \supset t)\}$ *(by (i) and since* $\Box(A \wedge B) \dashv\vdash_{\mathbf{K}} \{\Box A, \Box B\}$*)*

[11] As usual, we say that a logic **L** is at least as strong as a logic **L**$'$ iff for all $\Gamma \subseteq \mathcal{W}_{\mathbf{L}'}$, $Cn_{\mathbf{L}'}(\Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma)$.

[12] As explained in Section 2, we use $\vdash_{\mathbf{CL}}$ in a more general sense here, so that it can also range over richer languages.

(iii) $\Upsilon \dashv\vdash_{\mathbf{T}} \{p, \Box\neg q, \Box r, \sim s, \Box(s \vee t), \Box\Box(s \supset t)\}$ *(by (ii) and since* $\{A \vee B, \Box\neg B\} \dashv\vdash_{\mathbf{T}} \{A, \Box\neg B\}$*)*

(iv) $\Upsilon \dashv\vdash_{\mathbf{S4}} \{p, \Box\neg q, \Box r, \sim s, \Box t\}$ *(by (iii), since* $\vdash_{\mathbf{S4}} \Box A \equiv \Box\Box A$*, and by modal inheritance)*

In view of these facts, we can verify that each of the following holds:[13]

$\{\{p, q, r\}, \{s, t\}\}$ *is a* **CL**-*splitting of* Υ
$\{\{p, q\}, \{r\}, \{s, t\}\}$ *is a* **K**-*splitting of* Υ
$\{\{p\}, \{q\}, \{r\}, \{s, t\}\}$ *is a* **T**-*splitting of* Υ
$\{\{p\}, \{q\}, \{r\}, \{s\}, \{t\}\}$ *is a* **S4**-*splitting of* Υ

Although the proof for Theorem 2 is very short, this is a noteworthy result. Recall that in order to avoid that relevance results in triviality for inconsistent Υ, it was necessary to use subclassical splittings. Theorem 2 indicates that the stronger the paraconsistent logic of our choice, the better we may approximate the finest **CL**-splitting without ending up with triviality in the case of an inconsistency.

Another important consequence of Theorem 2 is the following:

Corollary 1. *Suppose that there is a finest* **L**-*splitting and a finest* **L**$'$-*splitting of* Υ. *If* **L** *is at least as strong as* **L**$'$ *and* B *is* **L**-*relevant to* Ψ *modulo* Υ, *then* B *is* **L**$'$-*relevant to* Ψ *modulo* Υ.

Conversely, if **L** is at least as strong as **L**$'$ and B is not **L**$'$-relevant to to Ψ modulo Υ, then B is also not **L**-relevant to to Ψ modulo Υ. So we obtain sufficient criteria to determine whether a certain belief is (ir)relevant to the current operation of belief change. This is particularly important since, as argued e.g. in [42], to compute the finest splitting of a certain theory (modulo **CL**) is a very arduous task; however, it may often suffice to show irrelevance at a lower level already, which may be significantly easier.

6. Some Technical Results

In this section, the finest splitting theorem from [25] and the least letter-set theorem from [27, 28] are generalized to a broad range of non-classical logics. Before doing so, I discuss some basic properties that are needed to get the generalization going, mentioning some examples of well-known logics that satisfy each of those properties. Eventually, this yields a partial answer to the question posed in the concluding section of [25]:

> […] how far can the results [of our paper] be established for sub-classical, (e.g. intuitionistic) consequence relations or supraclassical ones (e.g.,

[13] To avoid clutter, I will henceforth only mention the letters that are explicit in Υ.

preferential consequence relations or the relation of logical friendliness of Makinson [8])?

The proof of the existence of the finest **L**-splitting makes essential use of two ingredients: (i) the parallel interpolation theorem (see Section 6.5), which generalizes standard (Craig) interpolation and was shown for $\mathbf{L} = \mathbf{CL}$ in [25]; and (ii) the definition of a specific set $Min_{\mathbf{L}}(\Gamma)$, which has a number of interesting properties (see Section 6.2). Using (ii), we can also obtain a generalization and very elegant proof of the least letter-set theorem from [27], which is spelled out and discussed in Section 6.4.

6.1. *Some Basic Properties*

The following properties are very well-known, and require little explanation:

Reflexivity:	$\Gamma \subseteq Cn_{\mathbf{L}}(\Gamma)$
Transitivity:	if $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$, then $Cn_{\mathbf{L}}(\Gamma') \subseteq Cn_{\mathbf{L}}(\Gamma)$
Monotonicity:	$Cn_{\mathbf{L}}(\Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma \cup \Gamma')$
Compactness:	$\Gamma \vdash_{\mathbf{L}} A$ iff there are $B_1, \ldots, B_n \in \Gamma$ such that $\{B_1, \ldots, B_n\} \vdash_{\mathbf{L}} A$.
Standard Interpolation:	If $\Gamma \vdash_{\mathbf{L}} A$, then there is a B s.t. $\Gamma \vdash_{\mathbf{L}} B$, $B \vdash_{\mathbf{L}} A$, and $E(B) \subseteq E(\Gamma) \cap E(A)$.

If a logic has the first three properties, we say that it is a *Tarski-logic*.

For the proof of the parallel interpolation property (cf. supra, ingredient (i)) I will also need to rely on the assumption that the language of **L** contains an implication $\supset$ which satisfies two well-known requirements.[14]

Deduction Theorem:	If $\Gamma \cup \{A\} \vdash_{\mathbf{L}} B$, then $\Gamma \vdash_{\mathbf{L}} A \supset B$
Modus Ponens:	$\{A, A \supset B\} \vdash_{\mathbf{L}} B$

Let us briefly look at some well-known logics, to see whether these do or do not satisfy each of the above properties.

First of all, numerous normal modal logics satisfy them. Each of these have a classical implication, for which Modus Ponens is obviously valid. To obtain the Deduction Theorem for modal logics, we need to define the derivability relation $\vdash_{\mathbf{L}}$ in such a way that the necessity rule (where A is a theorem, to infer $\Box A$) can only be applied to formulas that have no premises in their path. This way of defining a "local" consequence relation for modal

[14] Of course, nothing excludes the existence of other implications besides $\supset$ – one may for instance have an additional relevant implication $\rightarrow$, like in the supraclassical relevant logics discussed in [29].

logics is fairly standard – see [19] for a lengthy discussion.[15] Finally, for the Interpolation property, we refer to [30] for an overview of some results in this area.

Let us now turn to sub-classical (compact) Tarski-logics. It is well-known that the above properties hold for intuitionistic logic **I**.[16] It was shown by Maksimova that only eight of the infinitely many intermediary logics (logics "between" **I** and **CL**) have the Interpolation property – see again [30] for a discussion of these results.

In [5], one may find the definition of nine compact Tarski-logics that are paraconsistent, paracomplete, or both.[17] As shown there, each of these logics satisfy Standard Interpolation. Moreover, each of them uses a classical implication, whence they have each of the above seven properties. One of these logics, viz. Schütte's system **CLuNs**, will be discussed in Section 7 below.

Some well-known subclassical systems fail to have some of the properties mentioned above. As already pointed out, only finitely many intermediary logics satisfy Standard Interpolation, which seems almost indispensable for the proof of the finest splitting theorem. Also, since in Priest's logic **LP** [35], $A \supset B =_{\text{def}} {\sim}A \vee B$, and since $\sim$ behaves paraconsistently, this logic does not satisfy Modus Ponens. I will mention some other examples in the concluding section. It seems plausible that the current results can be generalized to some of these, but the proofs may rely on more case-specific properties.

6.2. *The Set of L-minimal Formulas*

As noted, a crucial element in my proof of the finest splitting theorem is the definition of a unique set $Min_{\mathbf{L}}(\Gamma)$ for every Γ.[18] More precisely, I will use $Min_{\mathbf{L}}(\Gamma)$ in order to obtain an **L**-splitting of Γ, after which I show that this splitting is in fact the finest **L**-splitting of Γ.

Definition 7. *A is a minimal* **L**-*consequence of* Γ, $A \in Min_{\mathbf{L}}(\Gamma)$ *iff* $A \in Cn_{\mathbf{L}}(\Gamma)$ *and there is no* $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$ *such that (i)* $\Gamma' \vdash_{\mathbf{L}} A$ *and for every* $B \in \Gamma'$, $E(B) \subset E(A)$.

[15] The bottom line is that from an arbitrary premise A, we should of course not be able to derive $\Box A$ in a normal modal logic – otherwise $\Box$ becomes a meaningless operator. So one needs to distinguish between derivability of a formula from a given set of (contingent) premises (local derivability) and derivability of a theorem from the axioms of a modal logic (global derivability).

[16] See e.g. [12, Chapter 4] where this is shown for **I**.

[17] A logic **L** is paracomplete if $A \vee {\sim}A$ is not an **L**-theorem.

[18] The precise formulation in the definition of $Min_{\mathbf{L}}(\Gamma)$ benefited from a suggestion made by David Makinson (personal correspondence).

For instance, where $\Gamma = \{p \wedge r, {\sim}p \vee q, s \supset q, r \vee s\}$, we have that $p, r, q \in Min_{\mathbf{CL}}(\Gamma)$, but $p \wedge r, p \supset q, r \vee s \notin Min_{\mathbf{CL}}(\Gamma)$. More generally, the set $Min_{\mathbf{L}}(\Gamma)$ corresponds to the maximal level of analysis (in terms of the seperation of letters) the logic **L** allows us to perform. Note that this does not imply that each $A \in Min_{\mathbf{L}}(\Gamma)$ is also minimal in terms of logical symbols: e.g. if $p \in Min_{\mathbf{L}}(\Gamma)$ and the conjunction $\wedge$ behaves classicaly in **L**, then also $p \wedge p, p \wedge (p \wedge p), \ldots \in Min_{\mathbf{L}}(\Gamma)$.

$Min_{\mathbf{L}}(\Gamma)$ can be thought of as a generalization of the notion of prime implicates from the context of classical logic.[19] However, in view of the preceding paragraph, $Min_{\mathbf{L}}(\Gamma)$ is much larger than the set of prime implicates of Γ. Also, in the context of modal logics, the concept of prime implicates becomes problematic, as shown at length in [7]. In contrast, $Min_{\mathbf{L}}(\Gamma)$ is well-defined for arbitrary logics **L**.

Lemma 1. $Min_{\mathbf{L}}(\Gamma) \dashv\vdash_{\mathbf{L}} \Gamma$

Proof. In view of Definition 7, it suffices to prove the left-right direction. Suppose $A \in \Gamma$, whence by the reflexivity of **L**, $A \in Cn_{\mathbf{L}}(\Gamma)$. I prove by an induction that $A \in Cn_{\mathbf{L}}(Min_{\mathbf{L}}(\Gamma))$. If $A \in Min_{\mathbf{L}}(\Gamma)$, then by the reflexivity of **L**, $A \in Cn_{\mathbf{L}}(Min_{\mathbf{L}}(\Gamma))$. If $A \notin Min_{\mathbf{L}}(\Gamma)$, then since $A \in Cn_{\mathbf{L}}(\Gamma)$ and by Definition 7, there is a $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$, such that (i) $\Gamma' \vdash_{\mathbf{L}} A$ and (ii) for every $B \in \Gamma'$, $E(B) \subset E(A)$. For every $B \in \Gamma'$ such that $B \notin Min_{\mathbf{L}}(\Gamma)$, we repeat the same reasoning: since $B \in Cn_{\mathbf{L}}(\Gamma)$, there is a $\Gamma'' \subseteq Cn_{\mathbf{L}}(\Gamma)$ such that (i) for every $C \in \Gamma''$, $E(C) \subset E(B) \subset E(A)$ and (ii) $\Gamma'' \vdash_{\mathbf{L}} B$, whence by the transitivity and monotonicity of **L**, $(\Gamma' - \{B\}) \cup \Gamma'' \vdash_{\mathbf{L}} A$. Since A contains finitely many letters, we will at a finite point arrive at a set $\Delta \subseteq Min_{\mathbf{L}}(\Gamma)$ such that $\Delta \vdash_{\mathbf{L}} A$. By the monotonicity of **L**, $Min_{\mathbf{L}}(\Gamma) \vdash_{\mathbf{L}} A$. ■

Note that I only used the three Tarski-properties in the above proof; compactness and interpolation are not required.

6.3. *The Finest L-splitting*

In Section 6.5, I will generalize the parallel interpolation theorem from [25] to all logics **L** that satisfy each of the seven properties mentioned at the start of this section. However, to obtain the finest splitting theorem, it turns out that a slightly weaker property suffices, viz.

> *Non-compact Parallel Interpolation*: Let $\Delta = \bigcup_{i \in I} \{\Delta_i\}$ where the letter sets $E(\Delta_i)$ are pairwise disjoint, and suppose $\Delta \vdash_{\mathbf{L}} A$. Then there are sets

[19] A *prime implicate* of $\Gamma \subseteq \mathcal{W}_{\mathbf{CL}}$ is a minimal disjunction of literals that follows classically from Γ. See [42] and [8] where the notion of prime implicates is related to Parikh's axiom of relevance.

Θ_i such that (1) each $E(\Theta_i) \subseteq E(\Delta_i) \cap E(A)$, (2) each $\Delta_i \vdash_{\mathbf{L}} \Theta_i$, and (3) $\bigcup_{i \in I} \Theta_i \vdash_{\mathbf{L}} A$.

In this section, I presuppose that **L** is a Tarski-logic that has the Non-compact Parallel Interpolation property. Thus compactness need not be pre-supposed, and we need not make any assumptions about the connectives of **L**.

To obtain an **L**-splitting of Γ from the set $Min_{\mathbf{L}}(\Gamma)$, I first define a relation $\sim_\Delta$ over the members of Δ, for every $\Delta \subseteq \mathcal{W}_{\mathbf{L}}$:

Definition 8. *A is path-relevant to B modulo Δ ($A \sim_\Delta B$) iff there are $C_1, \ldots, C_n \in \Delta$ such that $E(A) \cap E(C_1) \neq \emptyset$, $E(C_1) \cap E(C_2) \neq \emptyset$, ..., and $E(C_n) \cap E(B) \neq \emptyset$.*

For instance, where $\Delta = \{p \vee q, r \vee q, r \vee s, t\}$, we can say that $p \sim_\Delta s$ and $p \wedge t \sim_\Delta s$, but $p \not\sim_\Delta t$. Note that in general, $\sim_\Delta$ can also hold between formulas that are not members of Δ.

On a historical side-note, $\sim_\Delta$ coincides with what Makinson calls *path-relevance* (modulo a set of formulas Δ) [28]. It was introduced by Rodrigues in his thesis [37] and also apppears the work of Wassermann, see e.g. [47].

It will be convenient to rely on the following property specific to $\sim_\Delta$ defined only over the members of Δ:

Fact 1. *$\sim_\Delta$ is transitive, reflexive and symmetric with respect to all A, B, $C \in \Delta$, whence $\sim_\Delta$ is an equivalence relation on the members of Δ.*

Definition 9. *$\mathbb{M}_{\mathbf{L}}(\Gamma)$ is the quotient set of $Min_{\mathbf{L}}(\Gamma)$ by $\sim_{Min_{\mathbf{L}}(\Gamma)}$.*[20] *Where $\mathbb{M}_{\mathbf{L}}(\Gamma) = \{\Delta_i\}_{i \in I}$, $\mathbb{E}_{\mathbf{L}}(\Gamma) = \{E(\Delta_i)\}_{i \in I} \cup \{\{A\} \mid A \in \mathcal{E} - E(Min_{\mathbf{L}}(\Gamma))\}$.*

Note that for no $\Delta \in \mathbb{M}_{\mathbf{L}}(\Gamma)$: $\Delta = \emptyset$, whence for no $E \in \mathbb{E}_{\mathbf{L}}(\Gamma)$: $E = \emptyset$. Also, in view of Definition 9, $\bigcup \mathbb{E}_{\mathbf{L}}(\Gamma) = \mathcal{E}$. To prove that $\mathbb{E}_{\mathbf{L}}(\Gamma)$ is a partition of $\mathcal{E}$, it thus suffices to show the following:

Lemma 2. *For every $E, E' \in \mathbb{E}_{\mathbf{L}}(\Gamma)$: $E \neq E'$ iff $E \cap E' = \emptyset$.*

Proof. Let $E, E' \in \mathbb{E}_{\mathbf{L}}(\Gamma)$. The right-left direction is obvious since no $E \in \mathbb{E}_{\mathbf{L}}(\Gamma)$ is empty. For the left-right direction, suppose that for $E, E' \in \mathbb{E}_{\mathbf{L}}(\Gamma)$, $E \cap E' \neq \emptyset$. I only consider the case where $E = E(\Delta)$ and $E' = E(\Delta')$ for Δ, $\Delta' \in \mathbb{M}_{\mathbf{L}}(\Gamma)$ – in the other case, it follows immediately that $E = E' = E \cap E' = \emptyset$. Suppose that $E(\Delta) \cap E(\Delta') \neq \emptyset$. This implies that there are $A \in \Delta$, $B \in \Delta'$: $E(A) \cap E(B) \neq \emptyset$, whence $A \sim_{Min_{\mathbf{L}}(\Gamma)} B$, hence A and B are in the same equivalence class. As a result, $\Delta = \Delta'$, whence $E = E'$. ■

[20] This is the set of all equivalence sets of $Min_{\mathbf{L}}(\Gamma)$, given the equivalence relation $\sim Min_{\mathbf{L}}(\Gamma)$ on $Min_{\mathbf{L}}(\Gamma)$.

Since $\sim_{Min_{\mathbf{L}}(\Gamma)}$ is an equivalence relation on $Min_{\mathbf{L}}(\Gamma)$, $\mathbb{M}_{\mathbf{L}}(\Gamma)$ is a partition of $Min_{\mathbf{L}}(\Gamma)$. Hence, from Lemma 1 and the fact that $\mathbb{E}_{\mathbf{L}}(\Gamma)$ is a partition of $\mathcal{E}$, we can infer that $\mathbb{E}_{\mathbf{L}}(\Gamma)$ is an $\mathbf{L}$-splitting of Γ. It remains to show that $\mathbb{E}_{\mathbf{L}}(\Gamma)$ is the *finest* $\mathbf{L}$-splitting of Γ:

Theorem 3. $\mathbb{E}_{\mathbf{L}}(\Gamma)$ *is the finest* $\mathbf{L}$*-splitting of* Γ.

Proof. Assume that there is a splitting $\mathbb{E}$ of Γ, such that $\mathbb{E}$ is finer than $\mathbb{E}_{\mathbf{L}}(\Gamma)$. Hence for some $E \in \mathbb{E}_{\mathbf{L}}(\Gamma)$, there is an $E' \in \mathbb{E}$: $\emptyset \subset E' \subset E$. This means that $E \neq \emptyset$, and hence $E = E(\Delta)$ for some $\Delta \in \mathbb{M}_{\mathbf{L}}(\Gamma)$. So we have:

(†) For a $\Delta \in \mathbb{M}_{\mathbf{L}}(\Gamma)$, there is an $E' \in \mathbb{E}$: $\emptyset \subset E' \subset E(\Delta)$

Let $A \in \Delta$ be such that $E(A) \nsubseteq E'$ and let $B \in \Delta$ be such that $E(B) \cap E' = \emptyset$. To see why A exists, assume that for every $A' \in \Delta$, $E(A') \subseteq E'$. In that case, $E(\Delta) \subseteq E'$, which contradicts (†). To see why B exists, assume that for every $B \in \Delta$, $E(B) \cap E' = \emptyset$. In that case, $E(\Delta) \cap E' = \emptyset$, which again contradicts (†).

Since $A, B \in \Delta$, $A \sim_{Min_{\mathbf{L}}(\Gamma)} B$. Hence there are $C_1, \ldots, C_n \in Min_{\mathbf{L}}(\Gamma)$ such that $E(A) \cap E(C_1) \neq \emptyset$, $E(C_1) \cap E(C_2) \neq \emptyset$, $E(C_2) \cap E(C_3) \neq \emptyset$, …, and $E(C_n) \cap E(B) \neq \emptyset$. Let $A = C_0$ and $B = C_{n+1}$.

Assume now that (‡) for every k with $0 \leq k \leq n + 1$, either $E(C_k) \cap E' = \emptyset$ or $E(C_k) \subseteq E'$. Then it can be shown by mathematical induction that

(*) for every k with $0 \leq k \leq n + 1$, $E(C_k) \cap E' = \emptyset$.

The base case ($k = 0$) is immediate, in view of (‡) and the fact that $E(A) \nsubseteq E'$. For the induction step, suppose that $E(C_k) \cap E' = \emptyset$. Since $E(C_k) \cap E(C_{k+1}) \neq \emptyset$, it follows that $E(C_{k+1}) \nsubseteq E'$. But then, in view of (‡), $E(C_{k+1}) \cap E' = \emptyset$.

From (*) and the fact that $B = C_{n+1}$, we can derive that $E(B) \cap E' = \emptyset$ – a contradiction. So assumption (‡) must be false: there is a k with $0 \leq k \leq n + 1$, such that $E(C_k) \cap E' \neq \emptyset$ and $E(C_k) \nsubseteq E'$. Let l be such that $E(C_l) \cap E' = \emptyset$ and $E(C_l) \nsubseteq E'$, and let $D = C_l$.

Since $\mathbb{E}$ is a splitting of Γ, $\{\bigcup \mathbb{E} - E', E'\}$ is also a splitting of Γ. Hence there are Θ, Θ' such that $\Theta \cup \Theta' \dashv\vdash_{\mathbf{L}} \Gamma$, $E(\Theta) \subseteq \bigcup \mathbb{E} - E'$ and $E(\Theta') \subseteq E'$. It follows that (♣) $E(\Theta) \cap E(\Theta') = \emptyset$. Moreover, since $\Gamma \vdash_{\mathbf{L}} D$, also $\Theta \cup \Theta' \vdash_{\mathbf{L}} D$, whence by (Non-Compact) Parallel Interpolation, there are two sets Λ and Λ' such that (1) $E(\Lambda) \subseteq E(\Theta) \cap E(D)$, (2) $E(\Lambda') \subseteq E(\Theta') \cap E(D)$ and (3) $\Lambda \cup \Lambda' \vdash_{\mathbf{L}} D$.

Since $\Theta \cup \Theta' \dashv\vdash_{\mathbf{L}} \Gamma$, also $\Gamma \vdash_{\mathbf{L}} \Lambda$ and $\Gamma \vdash_{\mathbf{L}} \Lambda$. By (♣), (1) and (2), $E(\Lambda) \subset E(D)$ and $E(\Lambda') \subset E(D)$. So for every $F \in \Lambda \cup \Lambda'$, $E(F) \subset E(D)$. Hence by (3), $D \notin Min_{\mathbf{L}}(\Gamma)$ — a contradiction. ■

6.4. *The Least Letter-set Theorem*

In this section, I generalize the least letter-set theorem to any Tarski-logic that satisfies the following variant of interpolation:[21]

> *Non-compact Standard Interpolation*: If $\Gamma \vdash_{\mathbf{L}} A$, then there is a Γ' such that (i) $\Gamma \vdash_{\mathbf{L}} \Gamma'$, (ii) $\Gamma' \vdash_{\mathbf{L}} A$ and (iii) $E(\Gamma') \subseteq E(\Gamma) \cap E(A)$

I refer to [28] for some more background on this theorem, and to [27, Appendix] for Makinson's (semantic) proof. Both papers are restricted to the case where $\mathbf{L} = \mathbf{CL}$.

However, we must be careful: the exact formulation of the theorem in [27] is slightly different from the one in [28], because it is applied in a different context.[22] My formulation is a variation on the one in [28]. The proof I will present is very short, thanks to the introduction of the concept of $\mathbf{L}$-minimality in the preceding section.

Theorem 4. *For every $\Gamma \subseteq \mathcal{W}_{\mathbf{L}}$, there is a unique $\Delta \subseteq \mathcal{E}$ such that (a) for every Γ' that is $\mathbf{L}$-equivalent to Γ: $\Delta \subseteq E(\Gamma')$ and (b) for a Γ'' that is $\mathbf{L}$-equivalent to Γ, $\Delta = E(\Gamma'')$. (Least Letter-set Theorem)*

Proof. Let $\Delta = E(Min_{\mathbf{L}}(\Gamma))$. (b) follows immediately by the construction and Lemma 1; hence it suffices to prove (a). Suppose (1) $\Gamma' \dashv\vdash_{\mathbf{L}} \Gamma$, but $\Delta \not\subseteq E(\Gamma')$. Hence there is an $A \in Min_{\mathbf{L}}(\Gamma)$: $E(A) \not\subseteq E(\Gamma')$, whence also (2) $E(A) \cap E(\Gamma') \subset E(A)$. By (1) and Definition 7, $\Gamma' \vdash_{\mathbf{L}} A$. By (non-compact) interpolation, there is a Θ such that (3) $\Gamma' \vdash_{\mathbf{L}} \Theta$, (4) $\Theta \vdash_{\mathbf{L}} A$ and (5) $E(\Theta) \subseteq E(\Gamma') \cap E(A)$. By (1) and (3), it follows that $\Theta \subseteq Cn_{\mathbf{L}}(\Gamma)$, and by (2) and (5), it follows that $E(B) \subset E(A)$ for all $B \in \Theta$. But then by (3) and in view of Definition 7, $A \notin Min_{\mathbf{L}}(\Gamma)$ — a contradiction. ■

6.5. *Parallel Interpolation for L*

In this section, I presuppose that $\mathbf{L}$ satisfies *all* seven properties mentioned in the preceding section. The proof for Theorem 5 is obtained through a variation on the proof for Theorem 1.1 in [25]. One crucial difference is that, where Kourousias and Makinson also rely on certain properties of the classical conjunction $\wedge$, I only use the aforementioned properties of $\supset$ to run the central argument of the proof.

[21] David Makinson pointed out to me that this weaker kind of interpolation suffices to run the proof.

[22] In [27], a specific set Γ^* is defined for every Γ, and it is shown that this set is a least letter-set representation of Γ. Γ^* is defined in semantic terms, and the proof proceeds likewise. On the other hand, the formulation of the least letter-set theorem in [28] is a "bare statement of existence" (Makinson, personal correspondence), without reference to any specific least letter-set representation.

Theorem 5. *Let $\Delta = \bigcup_{i\in I}\{\Delta_i\}$ where the letter sets $E(\Delta_i)$ are pairwise disjoint, and suppose $\Delta \vdash_{\mathbf{L}} A$. Then there are formulas B_i such that (1) each $E(B_i) \subseteq E(\Delta_i) \cap E(A)$, (2) each $\Delta_i \vdash_{\mathbf{L}} B_i$, and (3) $\bigcup_{i\in I}\{Bi\} \vdash_{\mathbf{L}} A$. (Parallel Interpolation)*

Proof. Suppose the antecedent holds. By the compactness of **L**, there is a finite subfamily of finite subsets of the Δ_i whose union implies A. Let these subsets be $\Theta_1, \ldots, \Theta_n$, and let for each $k \leq n$, $\Theta_k = \{B^k_1, \ldots, B^k_{mk}\}$.

So we have:

$$\{B^1_1, \ldots, B^1_{m_1}\} \cup \ldots \cup \{B^n_1, \ldots, B^n_{m_n}\} \vdash_{\mathbf{L}} A$$

By finitely many applications of the Deduction Theorem,

$$\{B^1_1, \ldots, B^1_{m_1}\} \vdash_{\mathbf{L}} B^2_1 \supset (B^2_2 \supset (\ldots \supset (B^n_{m_n} \supset A) \ldots))$$

By Standard Interpolation, there is a formula C_1 such that

(1.1) $\{B^1_1, \ldots, B^1_{m_1}\} \vdash_{\mathbf{L}} C_1$,

(1.2) $C_1 \vdash_{\mathbf{L}} B^2_1 \supset (B^2_2 \supset (\ldots \supset (B^n_{m_n} \supset A) \ldots)$, and

(2) $E(C_1) \subseteq E(\{B^1_1, \ldots, B^1_{m_1}\}) \cap E(B^2_1 \supset (B^2_2 \supset (\ldots \supset (B^n_{m_n} \supset A) \ldots)))$

However, note that the sets $E(\Theta_i)$ are pairwise disjoint. From this together with (2), we can infer:

$$E(C_1) \subseteq E(\{B^1_1, \ldots, B^1_{m_1}\}) \cap E(A)$$

and hence,

$$E(C_1) \subseteq E(\Theta_1) \cap E(A)$$

By (1.2), modus ponens and the monotonicity of **L**, we can derive:

$$\{C_1\} \cup \{B^2_1, \ldots, B^2_{m_2}\} \cup \ldots \cup \{B^n_1, \ldots, B^n_{m_n}\} \vdash_{\mathbf{L}} A$$

We may now repeat the same procedure, pushing all formulas $C_1, B^3_1, \ldots, B^n_{m_n}$ to the right side of the turnstile by means of the Deduction Theorem. From this, we obtain an interpolant C_2 such that $E(C_2) \subseteq E(\Theta_2) \cap E(A)$ and $\{C_1, C_2\} \vdash_{\mathbf{L}} B^3_1 \supset (B^3_2 \supset (\ldots \supset (B^n{}_{m_n} \supset A) \ldots))$. After n iterations of this reasoning, we have obtained $C_1, \ldots, C_n$, where each $E(C_j) \subseteq E(\Theta_j) \cap E(A)$, and $\{C_1, \ldots, C_n\} \vdash_{\mathbf{L}} A$. The rest is immediate in view of the construction. ■

6.6. *Summary of the Results*

The following three corollaries summarize the main results of the current section:

Corollary 2. *If* **L** *is a Tarski-logic that satisfies (Non-compact) Parallel Interpolation, then every $\Gamma \subseteq \mathcal{W}_{\mathbf{L}}$ has a finest* **L***-splitting.*

Corollary 3. *If* **L** *is a Tarski-logic that satisfies (Non-compact) Standard Interpolation, then every* $\Gamma \subseteq \mathcal{W}_{\mathbf{L}}$ *has a least letter-set representation in* **L**.

Corollary 4. *If* **L** *is a compact Tarski-logic that satisfies Standard Interpolation, and if one can define an implication* $\supset$ *in* **L** *that satisfies Modus Ponens and the Deduction Theorem, then* **L** *satisfies Parallel Interpolation.*

7. Example: CLuNs-relevance

7.1. *The Paraconsistent Logic CLuNs*

To explain the idea behind a non-classical relevance axiom, I will use the paraconsistent logic **CLuNs**, as axiomatized in [5]. The choice for this system is motivated by two properties of the logic:

(i) **CLuNs** is maximally paraconsistent, i.e. if we add an axiom to **CLuNs** that is not derivable in the logic, then we obtain full **CL**. As a result, the analytic power of **CLuNs** is very close to that of **CL**.

(ii) Nevertheless, **CLuNs** is also fully paraconsistent, i.e. there are no Γ such that $Cn_{\mathbf{CLuNs}}(\Gamma) = \mathcal{W}_{\mathbf{CLuNs}}$. Hence **CLuNs**-relevance will not trivialize any belief set.

Each of these advantages will be illustrated below. However, as explained in Section 5, the choice for a specific logic **L** as the underlying logic of a relevance axiom will depend on the specific application – this is not the place to argue for one logic in favor of others.

The propositional fragment of **CLuNs** is one of the three systems devised by Schütte in [38]. All three of these systems are particularly strong in that they allow us to drive the paraconsistent negation inwards; e.g. it is possible to derive $\sim A$, $\sim B$ from $\sim(A \vee B)$, and similarly to derive $A \wedge \sim B$ from $\sim(A \supset B)$. A distinctive feature of **CLuNs** is that it is paraconsistent but not paracomplete (unlike the other Schütte systems): it can model cases where both A and $\sim A$ are true, but it cannot model cases in which both are false.

The logic **CLuNs** is based on the language of **CL**. It is axiomatized by the rule **MP** (from A, $A \supset B$ to infer B), the axioms of full positive **CL**:

A⊃1 $A \supset (B \supset A)$

A⊃2 $((A \supset B) \supset A) \supset A)$

A⊃3 $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

A⊥ $\bot \supset A$

A∧1 $(A \wedge B) \supset A$

A∧2 $(A \wedge B) \supset B$

A∧3 $A \supset (B \supset (A \wedge B))$

A∨1 $A \supset (A \vee B)$
A∨2 $B \supset (A \vee B)$
A∨3 $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
A≡1 $(A \equiv B) \supset (A \supset B)$
A≡2 $(A \equiv B) \supset (B \supset A)$
A≡3 $(A \supset B) \supset ((B \supset A) \supset (A \equiv B))$

the rule of excluded middle:

EM $A \vee \sim A$

and the following axioms that drive negation inwards:

A∼∼ $\sim\sim A \equiv A$
A∼⊃ $\sim(A \supset B) \equiv (A \wedge \sim B)$
A∼∧ $\sim(A \wedge B) \equiv (\sim A \vee \sim B)$
A∼∨ $\sim(A \vee B) \equiv (\sim A \wedge \sim B)$
A∼≡ $\sim(A \equiv B) \equiv \sim(A \supset B) \vee \sim(B \supset A))$

Let $\Gamma \vdash_{\mathbf{CLuNs}} A$ iff there are $B_1, \ldots, B_n \in \Gamma$ such that $\vdash_{\mathbf{CLuNs}} (B_1 \wedge \ldots \wedge B_n) \supset A$.[23]

For reasons of space, I will not discuss the various semantic characterizations of **CLuNs** – see e.g. [5, 4, 45]. Note that since ⊃ and ⊥ behave classically in **CLuNs**, it is possible to define a classical negation ¬ in this system by $\neg A =_{\text{def}} (A \supset \bot)$.

CLuNs is an extension of Priest's logic **LP** [35] with the classical implication ⊃ (note that ⊃ can *not* be defined from ∼ and ∨ in **CLuNs**, since ⊃ satisfies detachment and since ∼ is paraconsistent in this logic). Alternatively, **LP** reduces to the ∼-∨- ∧-fragment of **CLuNs**.

To see how **CLuNs** behaves, consider $\Upsilon = \{\sim(p \supset (q \vee r)), (\sim s \vee (t \wedge \sim\sim u)) \wedge p, \sim(\sim q \wedge p), v, \sim v \wedge \sim q\}$. Each of the following holds:

(1) $\Upsilon \vdash$ **CLuNs** $p \wedge \sim(q \vee r)$ (by **A∼⊃**)
(2) $\Upsilon \vdash$ **CLuNs** $p, \sim q, \sim r$ (by (1) and **A ∧ 1**, **A ∧ 2**)
(3) $\Upsilon \vdash$ **CLuNs** $\sim s \vee (t \wedge u)$ (by **A ∧ 1** and **A∼∼**)
(4) $\Upsilon \vdash$ **CLuNs** $\sim s \vee t, \sim s \vee u$ (by (3) and **A ∧ 1**, **A ∧ 2**)
(5) $\Upsilon \vdash$ **CLuNs** $\sim\sim q \vee \sim p$ (by **A∼∧**)
(6) $\Upsilon \vdash$ **CLuNs** $q \vee \sim p$ (by (5) and *A*∼∼)
(7) $\Upsilon \vdash$ **CLuNs** $(p \wedge \sim p) \vee (q \wedge \sim q)$ (by (2), (6))

[23] It is also possible to define a natural deduction system for **CLuNs**, but this would merely distract us from the main purpose of the current paper.

Υ is clearly classically inconsistent. Since **CLuNs** invalidates disjunctive syllogisme, it is not possible to **CLuNs**-derive e.g. $\sim p$ from $\sim q$ and $q \vee \sim p$. Hence $\Upsilon \nvdash_{\mathbf{CLuNs}} \sim p$.

7.2. *CLuNs-relevance*

I will now illustrate the idea of non-classical relevance by means of **CLuNs**. Consider $\Upsilon' = \{p, \sim q, \sim r, \sim s \vee t, \sim s \vee u, \sim p \vee q, v, \sim v\}$. In view of (1)-(7), it follows immediately that $\Upsilon \vdash_{\mathbf{CLuNs}} \Upsilon'$. I leave it as an exercise to the reader to prove that also $\Upsilon' \vdash_{\mathbf{CLuNs}} \Upsilon$.

Note that Υ' can be partitioned into three subsets: $\Delta_1 = \{p, \sim q, \sim p \vee q\}$, $\Delta_2 = \{\sim r\}$, $\Delta_3 = \{\sim s \vee t, \sim s \vee u\}$ and $\Delta_4 = \{v, \sim v\}$. Note also that the sets $E(\Delta_i)$ ($i \in \{1, 2, 3, 4\}$) are pairwise disjoint. Hence $\mathbb{E}(\Upsilon) = \{\{p, q\}, \{r\}, \{s, t, u\}, \{v\}\}$ is a **CLuNs**-splitting of Υ.

Suppose that we contract Υ by $p \vee \sim s$. Note that $\{p, q\}$ and $\{s, t, u\}$ are the only sets Λ in $\mathbb{E}(\Upsilon)$ for which $\Lambda \cap E(p \vee \sim s) \neq \emptyset$. The axiom of **CLuNs**-relevance tells us the following: a formula $A \in Cn_{\mathbf{CLuNs}}(\Upsilon)$ is relevant to the $p \vee \sim s$ modulo Υ iff $E(A) \cap \{p, q\} = \emptyset$ or $E(A) \cap \{s, t, u\} = \emptyset$. Hence the following **CLuNs**-consequences of Υ are *not* relevant to $p \vee \sim s$ modulo Υ: $\sim r$, v, $\sim v$.

This immediately brings us to the axiom of relevance. In the current case, this axiom stipulates that the beliefs $\sim r$, v, $\sim v$ should be upheld. Note that this means that a contradiction has to be upheld, in order to obey $\mathbf{P}_{\mathbf{CLuNs}}$. However, the axiom does not require us to believe just anything: e.g. if we remove p from Υ', we obtain a non-trivial yet fairly rich belief set that does not **CLuNs**-entail $p \vee r$.

So, on the one hand, we are able to separate $\sim r$ from p, notwithstanding the fact that in the initial formulation of Υ, these formulas are tied to each other. On the other hand, some beliefs are still considered relevant to the new information, and removing some of these results in a reasonable contraction set. In short, we obtain a non-trivial, yet also non-trivializing relevance axiom for inconsistent belief sets.

One could ask oneself: should the beliefs v and $\sim v$ be upheld? If so, the resulting contraction set will remain inconsistent. But is this a sufficient reason to remove (either of) these beliefs from Υ? Clearly, they have little to do with the formula by which we are contracting, no matter whether we consider Υ in its initial formulation, or a more analysed version of it, such as Υ'.

According to the standard AGM approach, inconsistencies cannot occur in any contraction set. This also applies to the more recent approaches in terms of belief bases: in both cases, it is required that $\Upsilon \ominus A \nvdash_{\mathbf{CL}} A$. Hence any contradiction is removed from Υ whenever this set is contracted by a formula A. In contrast, if we combine the idea of a language splitting with a subclassical logical framework, we can model the intuition that our inconsistencies should be removed only locally. I return to this point in the next section.

8. Related Work

Paraconsistency and (Ir)relevance. In [36, p. 10], it is argued that a paraconsistent approach to belief revision allows us to model processes in which inconsistencies are removed one by one, such that the intermediary belief states remain inconsistent. The authors quote Fuhrmann, who writes the following in a section of his [14] titled "Local Inconsistency":

> [...] Thus, in the face of inconsistent theories we should want two things:
>
> *(a) localise inconsistencies* – an inconsistent theory should not be rendered totally corrupt just because some inconsistency has crept into the theory; and
>
> *(b) locally restore consistency* – we should be able to resolve one inconsistency at a time by contracting an inconsistent theory such that other inconsistencies, which we cannot yet resolve, may be carried over into the contraction theory.

In order to obtain (a) and (b), Fuhrmann recommends that "theories be generated from bases by means of a consequence operation induced by some paraconsistent logic." [14, p. 187]

Recall that in our example, the axiom stipulated that the inconsistency v, $\sim v$ is upheld, since it is not relevant to the formula $p \vee \sim s$ by which we ought to contract. More generally, the axiom of **CLuNs**-relevance does not distinguish between formulas that behave consistently and those that behave inconsistently; all that matters is whether formulas are relevant to the information that triggers the revision or contraction, modulo the initial belief state. If an inconsistency is *not* relevant in this sense, then it is upheld.

In Fuhrmanns terms, the axiom of **CLuNs**-relevance requires that we should restore consistency only locally. However, Fuhrmann still argues that one should "restore consistency as soon as one can" [14, p. 187]. On his view, whether or not we should remove a contradiction $A \wedge \sim A$, depends on the question whether we have clear evidence that favors A over $\sim A$ or vice versa, or that shows that neither A nor $\sim A$ rely on reliable information. The *Relevance* axiom is more abstract: the need to restore consistency is replaced by the need to solve any type of problem with our beliefs. It is also more precise than Fuhrmann's position: it uses a very exact notion of what it means that our evidence relates to $A \wedge \sim A$ (in terms of letter sets).

Parikh and Chopra: Multi-sets. In [9], a theory of belief revision is proposed in which the unit of change is a multi-set $\{\Upsilon_1, \ldots, \Upsilon_n\}$. Each Υ_i is closed under classical logic, but the various Υ_i are not put together into one single theory. This way, global inconsistency – the fact that $\bigcup_{i \leq n} \Upsilon_i$ is inconsistent – does not trivialize the belief state.

In this model, the sets $E(\Upsilon_i)$ need not be mutually disjoint, though in the principle case, it is assumed that they are at least distinct. The idea is that each Υ_i concerns a specific topic, yet some of them may be related to others. Chopra and Parikh also motivate their model in terms of minimal mutilation: when we perform a revision, we should only revise those theories Υ_i that are relevant to the new information, and leave all other sets Υ_j unchanged. Here, relevance is spelled out in a more direct way, i.e. as the overlap of letter sets $E(\Upsilon_i)$ and $E(A)$ where A is the new information.

Although this approach is well-motivated from the point of view of concrete applications, it seems to put the cart before the horse from a logical viewpoint. That is, the biggest advantage of the notion of a language splitting is that it is the *logic* itself that determines which parts of a theory are related to others. Our beliefs need not be "prepared" or "split" in any sense; whether or not a belief $B \in \Upsilon$ is **L**-relevant to $\Upsilon \bullet A$ depends solely on $Cn_{\mathbf{L}}(\Upsilon)$.

Relevance and Local Change. In their [24], Hansson and Wassermann have introduced "localized" versions of well-known operations of belief change. These include the following two [24, p. 51]:

> *Local Consolidation.* Inconsistencies are removed from some part of the belief base. The rest of the agent's beliefs may well be inconsistent. For instance, I can make my beliefs about biological evolution consistent, while retaining global inconsistency between biological and religious beliefs.
>
> *Local Revision.* A new belief is added to the belief base in such a way that a certain part of the resulting base is made (kept) consistent. If I see, for example, that it is a sunny day in Amsterdam, then this contradicts my belief that it is always raining in Holland, and leads to revision. This can be done without checking whether my beliefs about Brazilian politics are consistent with the new belief.

Hansson and Wassermann refer to the work of Parikh (see [24, p. 69]), but although they start from the same intuitions, the notion of local change is rather different from the idea of relevant belief change modulo a splitting. First and foremost, Hansson and Wassermann first localize the consequence relation $\vdash$, after which they "localize" both syntactic and semantic characterizations of belief change operations – they do not define a general relevance axiom. Second, it can be easily verified that their operations of local change do not always obey the *Relevance* axiom from the current paper. The main reason for this is that they are still highly syntax-sensitive. That is, to perform a local revision of Υ by A, one considers all consistent minimal subsets of Υ that either imply A or $\sim A$, and only revises beliefs B that occur in those subsets. Although this makes it possible to leave some inconsistencies unchanged, a lot still depends on the exact way we formalize our beliefs.

For instance, let $\Upsilon = \{p \wedge q, r, \sim r\}$ and let $A = \{\sim q\}$. Then if we "localize" the revision to $\sim q$, the inconsistent behaviour of r is simply ignored; the revision set will be $\{\sim q, r, \sim r\}$. On the other hand, should we "locally" revise $\Upsilon' = \{p, q, r, \sim r\}$ by $\sim q$, then the resulting revision set would be $\{p, \sim q, r, \sim r\}$, and hence p would be rescued.

9. In Conclusion

In this paper, it was shown how one can sensibly apply the relevance axiom and the related notion of a language splitting in a very broad range of contexts, beyond the traditional focus on classical theory revision. Upon inspection of the proofs from Section 6, the results easily generalize to the first order predicative level, with "elementary letters" understood as elementary predicate and function symbols.

As indicated before, the focus of this paper was on syntactic characterizations of belief change. It remains an open question how one can obtain semantic constructions that warrant certain forms of the relevance axiom in specific cases. The notion of a canonical form from [25, 28] could be crucial in such constructions.

A related problem concerns the compatibility of (variants of) **P** with existing postulates and axioms for operations of belief change. Recall that Parikh proved the compatibility of the standard AGM axioms for theory revision with his relevance axiom. As explained in Section 4, $\mathbf{P}_b$ is inconsistent with some well-known postulate for the revision of belief bases. In [31] it was shown that also in the context of belief update, Parikh's axiom cannot easily be combined with existing postulates (i.c. the KM-postulates for belief update). Finally, the AGM axioms for belief contraction are only consistent given certain minimal requirements on the underlying logic **L** of beliefs [13]. Similarly, one can ask which conditions on **L** warrant compatibility of a given set of postulates with **P**.

In view of the technical results from Section 6, one may also ask whether it is possible to further generalize the results from this paper to other non-classical (Tarski-)logics such as **LP** or Brazilian anti-intuitionistic logic.[24] To the best of my knowledge, no interpolation results are available yet for these systems. Moreover, for **LP**, Parallel Interpolation cannot be obtained from Standard Interpolation in the way it was done here, as the implication does not satisfy *Modus Ponens*.

Another interesting topic would be non-monotonic splittings. For instance, within the adaptive logic programme, quite a few systems have been developed that allow one to interpret a set of beliefs "as consistently as possible",

[24] See [35], resp. [2] for a characterization of these systems.

without trivializing inconsistent beliefs.[25] Some of these systems are equivalent to **CL** whenever the belief set is consistent, and most of them are usually much stronger than the existing monotonic and transitive paraconsistent logics. It would hence be worthwhile to see whether such non-monotonic logics also yield a finest splitting for every set – in that case, the associated relevance axiom would be very strong, but it would still not trivialize inconsistent belief sets.

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