Shrinking beta

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Shrinking beta

Abstract

Betas are used in many applications ranging from asset pricing tests, cost of capital estimation, investment management and risk management. Beta needs to be estimated, and to reduce estimation error, shrinkage to its cross-sectional average value of one is often applied. Since beta is the product of the return correlation of a security with the market and its relative return volatility to that of the market, we shrink correlation and volatility separately and evaluate its predictive power. We find economically and statistically significant gains from shrinking correlations more than volatilities.

Keywords: Beta, Correlation, Investing, Low-risk, Shrinkage, Volatility

JEL Classification: C01, C13, C58, G11, G17

1. Introduction

The Capital Asset Pricing Model (CAPM) of Sharpe (1964) presumes that the expected return on a security is solely and positively linearly related to its systematic risk, measured as the covariance of its returns with that of the market, also known as its beta. Although beta is not good in predicting returns, it can still be useful for a wide range of purposes such as security selection, risk management, investment fund performance evaluation, and capital budgeting decisions. All things equal, risk-averse investors should demand a higher expected return for stocks and investment projects with a higher beta. In any case, beta is an unobserved characteristic and needs to be estimated.

Hence, an important question for academics and investors is how to best estimate the beta of a security. It is especially important how to deal with estimation error. Fama and MacBeth (1973) sort stocks in beta portfolios and use these estimated *portfolio* betas that contain less noise as proxy for the beta of *individual* stocks. Vasicek (1973) generates a Bayesian estimate of beta that shrinks the empirical estimate towards one. Blume (1975) suggests a practical approach to combine the historical beta estimate with a prior of one in a ratio of two to one. These solutions reduce large biases at the most extreme beta estimates.

In this study, we focus on beta estimation using shrinkage techniques. We use methods that shrink the beta in its entirety to methods that reduce estimation error in correlation and volatility separately, and examine which best forecasts beta. We then apply these insights to investment management. In particular, we focus on construction of low-beta portfolios.

The main contribution of our study is that we empirically examine the effect of using different shrinkage parameters for the correlation and volatility when predicting equity betas. We then apply these insights to a specific application to construct low-risk portfolios and compare the ex-post portfolio characteristics of these methods.

Our empirical findings can be summarized as follows. First, shrinking correlations more than relative volatilities significantly improves beta forecasts compared to shrinking betas in their entirety. Second, investors can efficiently construct low-risk portfolios by combining regular unshrunk betas and volatilities, as this implicitly resembles portfolios based on shrunk beta estimates in which correlations are shrunk more than volatilities.

The remainder of this paper is structured as follows. Section 2 shortly discusses the data. In Section 3, we describe the methodology. Section 4 contains the empirical results on predicting betas. Our application to low-risk strategies is discussed in Section 5. Finally, Section 6 concludes.

2. Data

The data used in this research consist of the Chicago Research in Security Prices US stock data covering the period from 1 January 1963 to 31 December 2017. The weekly data is based on an aggregate of the returns from Monday to Friday. Only common shares that are traded on the New York Stock Exchange (NYSE), American Stock Exchange, and Nasdaq are included. Furthermore, at each point in time, penny-stocks and micro-caps are excluded, where penny-stocks are defined as stocks with a price equal to or below \$1. Following Fama and French (2008), micro-caps are defined as stocks from all stock exchanges that fall below the value of the 20th percentile of the market capitalization of the NYSE listed stocks. On average about 1500 large- and mid-cap stocks are included in the analysis. Figure A1 in the appendix shows the number of included stocks through time. The Treasury bill rate is from the online data library of Kenneth French, and used to calculate returns in excess of the risk-free rate.

3. Methodology

The market beta, which is the central measure of a security's expected return in the CAPM, is defined as the product of the stock *i*'s correlation with the market portfolio *m* and its volatility ratio relative to that of the market:

$$\beta_{i,t} = \frac{cov(R^e_{i,t}, R^e_{m,t})}{var(R^e_{m,t})} = \frac{\rho(R^e_{i,t}, R^e_{m,t}) \cdot \sigma(R^e_{i,t}) \cdot \sigma(R^e_{m,t})}{\sigma^2(R^e_{m,t})} = \rho(R^e_{i,t}, R^e_{m,t}) \cdot \frac{\sigma(R^e_{i,t})}{\sigma(R^e_{m,t})} \quad (1)$$

We assume that the correlations and volatility ratios are independent, such that the expectation of beta equals the product of the expectations of the two components, and for convenience ignore Jensen's inequality in the volatility ratio term.

Our approach to forecast betas only makes use of returns.[2] Estimating beta is commonly done using a rolling window with a certain look-back period and data frequency. The decomposition in Equation (1) allows us to estimate each of the three components separately to obtain an estimator for a security's market beta. Since the stocks in our data set form an unbalanced panel, we require that for the estimation window at least 50% of return data should be available, otherwise we set the beta estimate to a missing value and do not use the stock in portfolio construction for that particular point in time.

The obtained estimates of beta, correlation, and volatility may contain a substantial amount of estimation error that can be reduced by implementing shrinkage, although it is not a priori clear how much these estimates should be shrunk and to which target value.[3] However, instead of shrinking the beta estimate in its entirety to one as in Blume (1975), we propose to shrink the correlation and relative volatility separately to their cross-sectional mean, since both components may include a different degree of estimation error.

$$R_{i,m,t}^{shrink} = (1 - c) \cdot R_{i,m,t} + c \cdot R_{i,m,t}^{target} \tag{2}$$

$$V_{i,m,t}^{shrink} = (1 - v) \cdot V_{i,m,t} + v \cdot V_{i,m,t}^{target} \tag{3}$$

where $R_{i,m,t}^{shrink}$ is the shrunk correlation between the stock *i*'s returns and the market return based on the correlation estimate $R_{i,m,t}$. Furthermore, $V_{i,m,t}^{shrink}$ is the shrunk volatility ratio of the stock *i*'s return relative to that of the market with $V_{i,m,t}$ its estimate. The target values of the correlation component $R_{i,m,t}^{target}$ and relative volatility component $V_{i,m,t}^{target}$ are set equal to the

[2] Theoretical models such as those in Ehling and Heyerdahl-Larsen (2017) link time-varying stock correlations across good and bad states to preference heterogeneity and aggregate risk aversion, which we ignore here. We also do not include more sophisticated methods or other security information to estimate a security's beta, such as for example in Cosemans, Frehen, Schotman, and Bauer (2016).

[3] Welch (2019) proposes an alternative method to calculate robust security betas by capping outliers in individual stock returns.

cross-sectional mean over all securities at time t. It is important to note here that in case the correlations are shrunk more than the relative volatilities, this implicitly means that the spread between stock correlations is reduced more than the spread in volatilities. Such difference in shrinkage parameter implies that the importance of estimated correlations is reduced compared to estimated volatility.

4. Predicting betas

A security's market beta is an unobserved characteristic, which therefore has to be estimated. There are several choices to estimate these betas. To make sure that our results are not depending on one particular choice, we use a variety of sample lengths, which we label *forward* period, to distinguish these from the *look-back* period, which is the historical sample period used for the prediction of the betas, and a variety of data frequencies, ranging from daily to monthly. We start by examining which rolling-window look-back periods and data frequencies predict these differently estimated future betas best.

Table 1 shows the Mean Squared Error (MSEs) between the rolling-window estimates and their subsequent realized values for a range of forward periods, look-back periods, and data frequencies.[4] The three panels contain the MSE values for the correlation (Panel A), relative volatility (Panel B) and beta (Panel C) estimates. The minimum values per forward period and frequency (column) are highlighted with a grey background and text in bold. We see that in the majority of cases, the data frequency used in the forward-period is also the optimal frequency in the look-back period, but that the weekly frequency is sometimes also optimal for forward period frequencies that are daily or monthly. In most cases, the optimal look-back period is at least as long as the forward period. In the majority of cases, a five-year look-back period with weekly frequency gives the lowest MSE, especially when we are interested in predicting one-

[4] In the main text, we use the MSE that penalizes large differences more strongly, but we show in Tables A1 and A2 in the appendix that the Mean Absolute Deviation (MAD) would lead to qualitatively similar results.

to three-year weekly and monthly betas. This is consistent with Gilbert, Hrdlica, Kalodimos, and Siegel (2014), who show that daily beta estimates are not necessarily better risk measures.

Table 1: Predicting correlation, relative volatility, and beta

Forward and look-back periods range from 1 month (1M) to 5 years (5Y) and use daily (D), weekly (W), or monthly (M) data frequencies. Numbers in bold with grey background have the lowest mean squared error (MSE) for a forward period and frequency combination per column.

Panel A: MSE Correlation estimates

	Forward	**1M**	**6M**	**1Y**	**3Y**	**5Y**	**1Y**	**3Y**	**5Y**	**1Y**	**3Y**	**5Y**
Look-back	Freq	D	D	D	D	D	W	W	W	M	M	M
1M	D	0.0820	0.0545	0.0524	0.0539	0.0558	0.0740	0.0697	0.0699	0.1407	0.1072	0.1013
6M	D	0.0560	0.0247	0.0213	0.0219	0.0225	0.0388	0.0333	0.0332	0.1027	0.0677	0.0616
1Y	D	**0.0547**	**0.0224**	**0.0192**	0.0190	0.0195	0.0356	0.0293	0.0290	0.0991	0.0628	0.0564
3Y	D	0.0562	0.0231	0.0194	**0.0174**	**0.0175**	0.0342	0.0257	0.0251	0.0955	0.0570	0.0497
5Y	D	0.0572	0.0239	0.0202	0.0178	0.0184	0.0347	0.0260	0.0256	0.0950	0.0558	0.0489
1Y	W	0.0732	0.0382	0.0341	0.0324	0.0320	0.0414	0.0334	0.0322	0.0941	0.0560	0.0489
3Y	W	0.0683	0.0325	0.0281	0.0241	0.0239	0.0335	0.0228	0.0220	0.0844	0.0431	0.0361
5Y	W	0.0674	0.0314	0.0269	0.0232	0.0232	**0.0322**	**0.0219**	**0.0211**	**0.0820**	**0.0411**	**0.0341**
1Y	M	0.1395	0.1018	0.0974	0.0967	0.0969	0.0939	0.0861	0.0856	0.1324	0.0942	0.0875
3Y	M	0.1084	0.0700	0.0654	0.0610	0.0613	0.0586	0.0469	0.0468	0.0949	0.0519	0.0456
5Y	M	0.1028	0.0640	0.0591	0.0550	0.0557	0.0523	0.0412	0.0411	0.0876	0.0450	0.0389

Panel B: MSE Relative Volatility Estimates

	Forward	**1M**	**6M**	**1Y**	**3Y**	**5Y**	**1Y**	**3Y**	**5Y**	**1Y**	**3Y**	**5Y**
Look-back	Freq	D	D	D	D	D	W	W	W	M	M	M
1M	D	3.8839	2.5870	2.4854	2.6733	2.8259	3.2655	3.4107	3.5528	4.2146	4.1976	4.2800
6M	D	2.8172	1.1733	0.9959	1.0312	1.1031	1.3994	1.4350	1.5389	2.1362	2.0210	2.0724
1Y	D	**2.7354**	**1.0620**	**0.8515**	0.8424	0.9013	1.1434	1.1557	1.2505	1.8621	1.6856	1.7272
3Y	D	2.8790	1.0928	0.8598	0.7450	0.7982	1.0069	0.9456	1.0319	1.6175	1.3965	1.4134
5Y	D	2.8742	1.1284	0.9002	0.7743	0.7983	1.0208	0.9554	1.0115	1.6022	1.3701	1.3664
1Y	W	3.3964	1.3833	1.0698	0.8592	0.8260	0.8951	0.7726	0.7916	1.2995	1.0131	0.9975
3Y	W	3.4565	1.3584	1.0343	0.7509	0.7253	0.7548	0.5759	0.6049	1.0689	0.7557	0.7398
5Y	W	3.3723	1.3202	1.0053	**0.7406**	**0.6928**	**0.7222**	**0.5640**	**0.5705**	1.0264	0.7269	0.6984
1Y	M	4.3927	2.1324	1.7491	1.4406	1.3730	1.2624	1.0715	1.0716	1.4301	1.0809	1.0554
3Y	M	4.2907	1.9667	1.5725	1.1998	1.1328	0.9843	0.7366	0.7370	1.0513	0.6885	0.6486
5Y	M	4.1615	1.8934	1.5123	1.1530	1.0513	0.9208	0.6875	0.6538	**0.9900**	**0.6198**	**0.5606**

Panel C: MSE Beta Estimates

	Forward	**1M**	**6M**	**1Y**	**3Y**	**5Y**	**1Y**	**3Y**	**5Y**	**1Y**	**3Y**	**5Y**
Look-back	Freq	D	D	D	D	D	W	W	W	M	M	M
1M	D	1.2180	0.7071	0.6691	0.6642	0.6846	0.7959	0.7347	0.7383	1.1541	0.8469	0.8180
6M	D	0.7421	0.2116	0.1771	0.1725	0.1820	0.2759	0.2196	0.2221	0.6229	0.3259	0.2955
1Y	D	0.7162	0.1847	0.1483	0.1360	0.1425	0.2409	0.1778	0.1779	0.5783	0.2805	0.2489
3Y	D	0.7046	**0.1823**	**0.1402**	**0.1138**	0.1152	0.2193	0.1463	0.1431	0.5415	0.2414	0.2093
5Y	D	**0.6983**	0.1882	0.1451	0.1139	**0.1133**	0.2174	0.1429	0.1388	0.5261	0.2324	0.2014
1Y	W	0.8118	0.2687	0.2276	0.2070	0.2068	0.2953	0.2264	0.2214	0.6158	0.3138	0.2794
3Y	W	0.7404	0.2104	0.1663	0.1348	0.1326	0.2245	0.1475	0.1419	0.5311	0.2294	0.1965
5Y	W	0.7239	0.2062	0.1614	0.1265	0.1220	**0.2148**	**0.1374**	**0.1305**	**0.5099**	**0.2151**	**0.1829**
1Y	M	1.2155	0.6486	0.5875	0.5408	0.5241	0.6384	0.5412	0.5228	0.9397	0.6074	0.5587
3Y	M	0.8570	0.3216	0.2747	0.2376	0.2338	0.3163	0.2337	0.2283	0.6023	0.2947	0.2638
5Y	M	0.8055	0.2836	0.2376	0.2021	0.1980	0.2771	0.1986	0.1932	0.5518	0.2578	0.2286

Panel A and B show that this longer-term lookback period is most efficient for both components of beta: Correlation and volatility. Intuitively, one could argue that correlations move more slowly than relative volatilities. However, we find no empirical support to differentiate the look-back period for correlations and volatilities.[5] For ease of presentation the remainder of this paper we use a look-back period of five years using weekly data for correlations and volatilities. This setting is robust and can predict betas on a one- to five-year basis. Hence, for evaluation of betas, we use a three-year forward horizon with weekly frequency. This horizon connects well with our application, as Van Vliet (2018) finds evidence that the optimal holding of a stock in a low-risk portfolio when transactions costs are taken into account is close to three years.[6]

Table 2 shows the effect of shrinkage on the prediction error. Panel A contains two-parameter shrinkage results, with volatility on the horizontal axis and correlation on the vertical axis, while Panel B contains the prediction error when only one shrinkage factor is applied. No shrinkage is denoted with 0, where the no shrinkage value of 0.1374 is also displayed in Table 1C. The optimal shrinkage factors for the sample estimates are $c = 0.5$ (correlation shrinkage) and $v = 0.2$ (volatility shrinkage). These optimal factors result in a MSE value of 0.1121.

Table 2 Panel B contains the results of four important statistical tests. The difference between the two-parameter shrinkage and the no shrinkage estimate is statistically significant using the Diebold and Mariano (1995) test statistic (DM = 5.92).[7] The MSE of the optimal two-parameter shrinkage estimate compared to one-parameter shrinkage of 0.2 or 0.5 on the diagonal is statistically significantly lower (DM = 3.68 and DM = 2.17, respectively). Unlike the look-back period, which is similar for correlations and relative volatilities, optimal shrinkage is different for these two components of the beta. The differences between optimal one- and two-parameter shrinkage is statistically significant at the 10% level using the Diebold and Mariano (1995) test

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[5] For example, Frazzini and Pedersen (2014) use a one-year look-back period for volatility and a five-year look-back period for correlation.

[6] With a longer holding period, we also circumvent possible "beta bubbles", i.e. mean-reversion in betas, as documented by Jylhä, Suominen, and Tomunen (2018). For more on estimating high-frequency betas, see Hansen, Lunde, and Voev (2014).

[7] We use the Newey and West (1987) estimator with 36-month lag to deal with the overlapping nature of our estimates.

statistic (DM = 1.63). Panel A shows an optimal shrinkage factor of 0.3 in case we only use one single shrinkage factor for correlation and volatility alike, similar to the one-third that Blume (1975) suggested.[8]

Table 2: Comparison of two-parameter and one-parameter shrinkage

Look-back period is five-years weekly data. Forward period is three years weekly data. Panel A contains two-parameter shrinkage, with correlation vertically and volatility horizontally. Diagonal has grey background, minimum MSE has darker background and is in bold. No shrinkage is denoted with 0. Panel B contains the Diebold and Mariano (1995) test for differences in prediction accuracy

Panel A: MSE of rolling-window beta estimate with two shrinkage factors

c↓ v⟶	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.1374	0.1327	0.1304	0.1304	0.1327	0.1374	0.1443	0.1536	0.1652	0.1791	0.1954
0.1	0.1304	0.1259	0.1237	0.1239	0.1263	0.1311	0.1382	0.1475	0.1592	0.1733	0.1896
0.2	0.1250	0.1207	0.1186	0.1188	0.1214	0.1262	0.1334	0.1429	0.1547	0.1688	0.1852
0.3	0.1212	0.1169	0.1149	0.1152	0.1178	0.1228	0.1300	0.1396	0.1515	0.1657	0.1822
0.4	0.1190	0.1147	0.1127	0.1131	0.1158	0.1208	0.1281	0.1377	0.1497	0.1640	0.1806
0.5	0.1184	0.1141	**0.1121**	0.1124	0.1151	0.1202	0.1275	0.1373	0.1493	0.1637	0.1805
0.6	0.1193	0.1149	0.1129	0.1132	0.1159	0.1210	0.1284	0.1382	0.1503	0.1648	0.1817
0.7	0.1218	0.1173	0.1152	0.1155	0.1182	0.1232	0.1307	0.1405	0.1527	0.1673	0.1843
0.8	0.1259	0.1213	0.1191	0.1193	0.1219	0.1269	0.1344	0.1442	0.1565	0.1713	0.1884
0.9	0.1316	0.1268	0.1244	0.1245	0.1270	0.1320	0.1395	0.1494	0.1617	0.1766	0.1939
1	0.1388	0.1338	0.1312	0.1312	0.1336	0.1385	0.1460	0.1559	0.1684	0.1833	0.2007

Panel B: Statistical tests	**DM**	**p-val**
No shrinkage versus optimal two-parameter shrinkage	5.92	0.000
Optimal two-parameter versus symetric 0.2/0.2 parameter shrinkage	3.68	0.000
Optimal two-parameter versus symetric 0.5/0.5 parameter shrinkage	2.17	0.015
Optimal two-parameter versus optimal one-parameter shrinkage	1.63	0.052

To further test if two-parameter shrinkage improves beta estimates, we perform an additional analysis that zooms in on stocks for which the beta estimates between two methods differ most. Beta estimates that are similar among methods are not that relevant observations to use in comparing which model leads to better forecasting performance. We now rank stocks on the *difference* in the beta estimates between two methods and form 10 portfolios on this difference. To examine whether there is an asymmetric forecasting performance relationship, we rank on the differences and not the absolute value of the differences. We then compare the average *ex-*

[8] This is also close to optimal value reported in Frazzini and Pedersen (2014) propose to shrink beta (with a mix of look-back periods for volatility and correlation) to the cross-sectional mean (one) with a shrinkage factor of 0.4. For this sample we also replicate the Frazzini and Pederson (2014) parameter settings: one-year look-back period for volatility and a five-year look-back period for correlation and a shrinkage factor for beta of 0.4. This results in a MAD value of 0.1241.

ante beta estimates with the realized betas over the full sample period. As we do not differentiate forecasting performance between under- or overestimation, the average absolute forecast error is used to gauge its empirical performance.

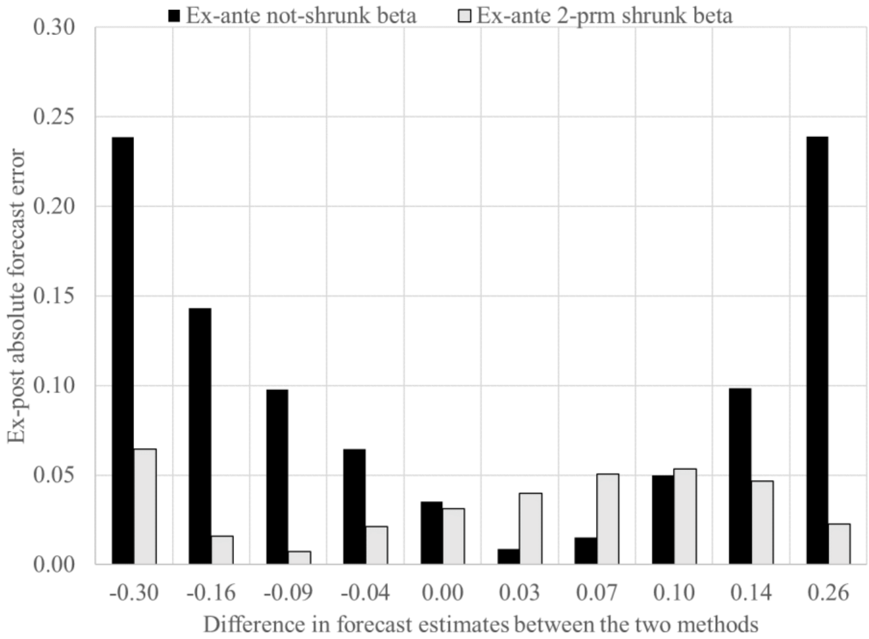
Figure 1 contains these forecasting errors. Panel A compares the rolling-window estimator without shrinkage and the two-parameter shrinkage that we propose in this paper, while Panel B compares the one- and two-parameter shrinkage methods with each other. The horizontal axis contains the difference in average beta estimates for each of the 10 decile portfolios that are created based on these differences. Both Panel A and Panel B clearly show that the forecasting errors are lower for the two-parameter shrinkage method.

It also becomes clear that the two-parameter shrinkage method is much better when it matters most. For example, Panel A shows that for the D10 portfolio the ex-ante difference in beta estimate is 0.26 and the ex-post difference with the realized beta is 0.24 for the rolling window estimator and 0.02 with the two-parameter shrinkage method. In Panel B the ex-ante difference for the D10 portfolio is 0.19, and the one-parameter shrinkage method has a forecast error of 0.19, while this is only 0.01 for the two-parameter shrinkage. For portfolios where the ex-ante methods do not differ much, the forecast errors are low and forecasting betas for those cases does not seem to be that challenging to begin with.

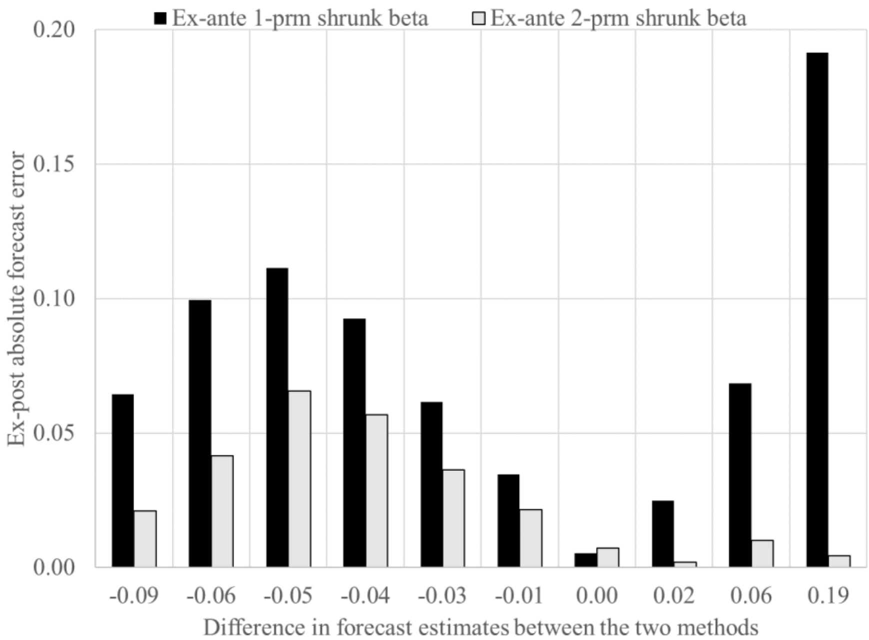
Figure 1: Comparing prediction errors where it matters most

Ex-post absolute forecast error of 10 portfolios each containing 10% of stocks ranking on <u>*differences*</u> *in estimated betas, with D1 contains the stocks for which the not-shrunk beta (Panel A) or one-parameter shrunk beta (Panel B) are larger than the two-parameter shrunk betas. The vertical axis contains the absolute value of the difference between the average ex-ante estimate and the realization over the entire sample period. The numbers on the horizontal axis are the differences between ex-ante estimates for each of the 10 portfolios.*

Panel A: Portfolios sorted on the difference in not-shrunk and 2-parameter shrinkage



Panel B: Portfolios sorted on the difference in 1- and 2-parameter shrinkage



5. Practical application to low-risk portfolios

We could stop our analysis after the previous section, as we established that betas can be more accurately estimated by two-parameter than one-parameter shrinkage. However, we also want to show how practitioners may use these superior beta estimates in portfolio management of low-risk portfolios. Empirically, CAPM beta is unable to explain returns since the risk-return relation is too flat or even negative.[9] This has led academics and investors to focus on constructing low-risk portfolios. In our application, we use the portfolio ranking methodology, i.e. we separate all stocks at the end of each month in 10 portfolios based on their beta, and calculate the 10 equally-weighted portfolio returns going forward.[10] Note that this sorting algorithm based on unshrunk betas or betas that are shrunk to one with the same shrinking factor lead to exactly the same portfolio, as the order of betas will not be affected by such transformation. However, when we use a different shrinking factor for the correlation and volatility, the order of betas may change and hence portfolios likely contain different stocks than those sorted on unshrunk betas. In this section, we aim to improve low-risk portfolio management by applying the insights from the previous sections.

5.1 Comparing beta-sorted portfolios

Black, Jensen, and Scholes (1972) show that low beta stocks tend to have a positive CAPM alpha. As shown above, the two-parameter shrunk beta estimate is a better prediction of the forward beta. We now turn to asking whether the improved *ex-ante* beta estimates at the individual stock level also led to lower *ex-post* portfolio betas.

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[9] See, Black, Jensen, and Scholes (1972), Haugen and Baker (1991), Clarke, De Silva, and Thorley (2006), Blitz and Van Vliet (2007), Baker, Bradley, and Wurgler (2011), Chow, Hsu, Kuo, and Li (2014), Frazzini and Pedersen (2014), and Blitz, Van Vliet, and Baltussen (2020). Blitz, Falkenstein, and Van Vliet (2014) review a wide range of explanations for this low-risk effect.

[10] Soe (2012) finds that minimum-variance portfolio optimization leas to a similar degree of risk reduction, even though the ranking approach typically ignores correlations between stock returns. For more details on estimating high-frequency covariance matrices that can accommodate trade asynchronicities, see, e.g., Boudt, Laurent, Lunde, Quaedvlieg, and Sauri (2017). Callot, Kock, and Medeiros (2017) use Lasso-type estimators to reduce the dimensionality of estimating large covariance matrices.

Table 3: Comparing the characteristics of beta-ranked portfolios

Ex-post portfolio characteristics of 10 portfolios each containing 10% of stocks ranking on estimated historical beta, with D1 lowest beta and D10 the highest beta. The excess returns are relative to the risk-free rate. D1-D10 is a long position in portfolio D1 and a short position in D10. Panel A contains the statistics when sorted on historical rolling-window betas (look-back 5 years with weekly returns). In Panel B, we apply fixed two-parameter shrinkage (v = 0.2 and c= 0.5).

Panel A: Portfolios based on historical rolling-window beta

	D1	D2	D3	D4	D5	D6	D7	D8	D9	D10	D1-D10
Excess return (%)	7.26	9.00	9.89	9.97	10.17	9.68	9.53	9.12	8.53	7.26	0.00
Standard dev. (%)	11.04	12.70	14.30	15.49	16.70	18.17	19.70	22.03	25.55	32.59	28.05
Sharpe ratio	0.66	0.71	0.69	0.64	0.61	0.53	0.48	0.41	0.33	0.22	0.00
CAPM alpha (%)	3.41	3.81	3.74	3.20	2.78	1.56	0.61	-0.90	-2.89	-6.74	10.14
t-statistic	3.45	4.73	5.26	4.63	4.24	2.48	1.14	-1.68	-3.34	-4.15	4.29
CAPM beta	0.45	0.61	0.72	0.80	0.87	0.96	1.05	1.18	1.34	1.65	-1.19

Panel B: Portfolios based on two-parameter shrinkage beta

	D1	D2	D3	D4	D5	D6	D7	D8	D9	D10	D1-D10
Excess return (%)	7.48	9.03	9.58	9.72	10.37	9.68	9.69	9.95	8.71	6.22	1.26
Standard dev. (%)	10.79	12.64	14.18	15.65	16.71	18.17	19.95	22.20	26.00	33.18	29.30
Sharpe ratio	0.69	0.71	0.68	0.62	0.62	0.53	0.49	0.45	0.33	0.19	0.04
CAPM alpha (%)	3.89	3.93	3.55	2.92	3.03	1.57	0.65	-0.13	-2.87	-7.98	11.87
t-statistic	3.81	4.69	4.70	4.00	4.26	2.44	1.22	-0.24	-3.08	-4.70	4.82
CAPM beta	0.42	0.60	0.71	0.80	0.86	0.95	1.06	1.19	1.36	1.67	-1.25

Table 3 shows the decile portfolio statistics based on ranking stocks on their historical beta estimate in Panel A and their two-parameter shrinkage beta estimate in Panel B. In addition to the decile portfolios, we also include a column with a long position in the D1 portfolio and a short position in the D10 portfolio, which combines the positive alpha from D1 with the negative alpha of D10.

Panel A shows that the decile with the lowest historical volatility, D1, has an average excess return of 7.26%, volatility of 11.04%, Sharpe ratio of 0.66, and market beta of 0.45. Note that the beta here is the ex-post beta of the decile portfolios, and not the ex-ante beta on which the sorting is based. The decile with the highest historical volatility, D10, also has an average return of 7.26%, but instead a volatility of 32.59%, Sharpe ratio of 0.22, and a market beta of 1.65. The D1-D10 portfolio has an alpha of 10.14% relative to the CAPM. This is the low-beta effect originally documented by Black, Jensen, and Scholes (1972).

Panel B uses the two-parameter (i.e. correlation and relative volatility separately) shrinkage method that we have shown above predicts future betas better than the estimates without shrinkage or that shrink beta in its entirety. Note that one-parameter shrinkage on the beta with the same shrinkage parameter for each stock does not change the ordering of betas, and hence the results from the portfolio sorts will be the same as for the beta estimates without shrinkage presented in Panel A. The shrinkage parameters are set equal to $c = 0.5$ and $v = 0.2$, based on its ability of best forecasting betas. We see that this two-parameter shrinkage leads the D1 portfolio to be less risky both in volatility as well as beta terms, and the D10 portfolio riskier. The volatility of the D1 portfolio declines from 11.04% to 10.79% and its beta from 0.45 to 0.42, while for D10 the volatility increases from 32.59% to 33.18% and its beta from 1.65 to 1.67. However, these differences are not statistically significant. The alpha in the D1-D10 portfolio increases from 10.14% to 11.87%. This increase is statistically significant with a p-value of 0.00 and the increase of the Sharpe ratio from 0.55 to 0.60 is also statistically significant with a p-value of 0.01 based on the Jobson and Korkie (1983) test with Memmel (2003) correction.

5.2 Comparing low-beta and low-volatility portfolios

Blitz and Van Vliet (2007) document both a positive alpha for low-volatility and low-beta portfolios, where alphas of low-volatility are generally higher. Here, we found that beta predictions are more accurate when correlations are shrunk more to their cross-sectional average than relative volatilities. An interpretation of our result is that estimated correlations receive less weight in the portfolio construction than estimated relative volatilities, because individual estimated correlations are shrunk more to the cross-sectional mean correlation than relative volatilities. We now take a different approach and form portfolios based on two-parameter shrunk betas and form portfolios based only on volatility, and form portfolios on combinations of these two variables. Because the beta and volatility estimates have a different mean and variance, we standardize them to obtain a joint ranking:

$$Z_{ms,t}^{combined} = (1 - q) \times Z(\beta_t) + q \times Z(\sigma_t) \quad (4)$$

where β_t are the (shrunk) beta and σ_t volatility estimates and Z(.) denotes cross-sectional robust z-scores.[11] For $q = 0$ the beta estimate is as in the previous subsection, whereas for $q = 1$ we have a pure volatility strategy in which correlations are completely ignored. This analysis gives insight into the effect of including correlations in addition to only volatility, and therefore connects the literature of low-beta and low-volatility investing.

Table 4: The impact of including correlations on volatility

Annualized volatility of the D1 portfolio for different forward and frequency combinations with optimal look-back and frequencies displayed in Table 1. For q=0 we have the standard two-parameter shrunk beta portfolio, while for q=1 we have a rank based purely on a stock's volatility. The lowest volatility by row is highlighted with shaded background and bold numbers.

Volatility (%)

Forward	↓Freq → ***q***	**0.0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
1M	D	10.84	10.79	10.69	10.68	10.64	10.64	10.62	**10.62**	10.65	10.68	10.74
6M	D	10.70	10.70	10.66	10.62	10.63	10.63	**10.62**	10.63	10.64	10.69	10.74
1Y	D	10.67	10.63	10.63	10.62	10.63	10.62	**10.61**	10.65	10.68	10.71	10.74
3Y	D	10.94	10.86	10.86	**10.81**	10.82	10.83	10.84	10.84	10.91	11.00	11.06
5Y	D	11.33	11.26	11.18	11.09	10.99	10.96	10.92	10.86	**10.83**	10.89	10.98
1Y	W	10.79	**10.75**	10.79	10.79	10.80	10.77	10.77	10.82	10.83	10.89	10.98
3Y	W	10.79	**10.75**	10.79	10.79	10.80	10.77	10.77	10.82	10.83	10.89	10.98
5Y	W	10.86	10.80	10.78	10.78	10.79	10.80	**10.78**	10.79	10.81	10.88	10.98
1Y	M	10.87	10.86	10.85	10.86	10.86	**10.83**	10.85	10.91	10.89	10.97	11.01
3Y	M	10.85	10.84	10.85	10.87	**10.83**	10.85	10.87	10.88	10.92	10.96	11.01
5Y	M	10.86	10.85	10.84	10.85	10.86	**10.84**	10.87	10.91	10.91	10.97	11.01

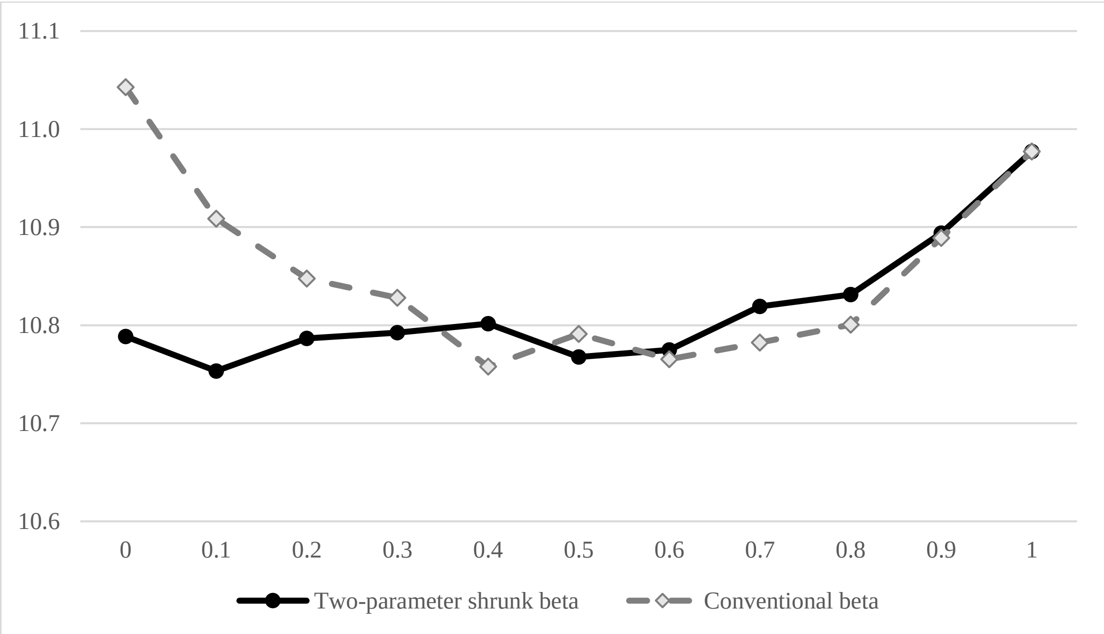
Table 4 shows the volatilities for the D1 portfolio, for a varying parameter q, based on the shrunk beta and a pure volatility estimate. For the forward period of 3 years with weekly frequency (with optimal look-back period of 5 years with weekly frequency), the volatility of 10.79% in Panel A corresponds to the volatility of the D1 portfolio reported in the previous table. Note that the vertical axis contains the forward-period and forward-frequency, which may lead to the same historical estimation period and frequency (see Table 1), hence resulting in duplicate rows when the optimal shrinkage parameters are also the same. Table 4 shows that

[11] Compared to regular z-scores, we replace the (cross-sectional) average with the median and the (cross-sectional) standard deviation with a constant (1.483) times the median absolute deviation. The normalized values are capped at -3 and 3 to further reduce the impact of outliers. See Rousseeuw and Croux (1993) for more on robust z-scores.

the majority of the lowest D1 volatilities are achieved for parameter q ranging between 0.3 to 0.7.

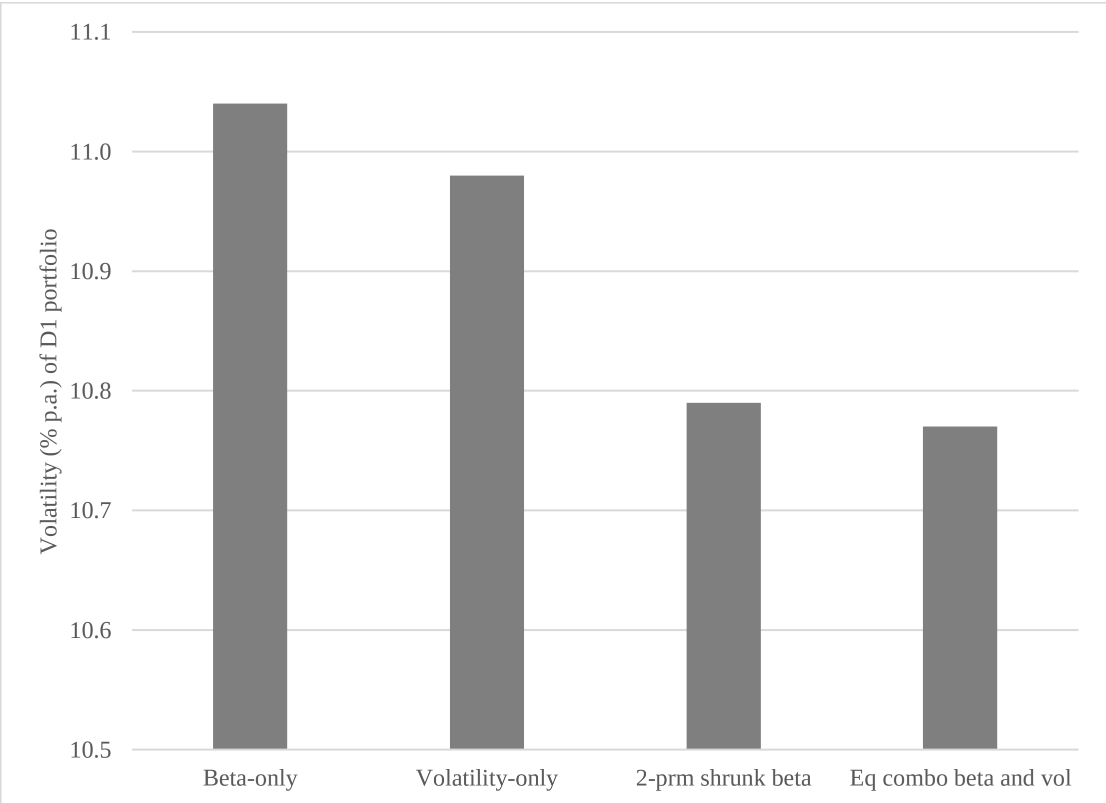
For the weekly and monthly forward frequencies, the optimal volatilities are significantly lower than the border case $q = 1$ at a 10% confidence level, which indicates that including correlation significantly reduces the volatility of a low-risk investment portfolio. In all cases except one, the lowest volatilities are not significantly lower than the $q = 0$ border case. This suggests that adding more volatility does not help to significantly reduce the volatility of a low-beta strategy, provided that the correlations are shrunk more than the volatilities when estimating a security's beta.

Figure 2: Comparing beta-ranked and volatility-ranked portfolios
Annualized volatility of the D1 portfolio for 3 year weekly forward period and frequency with 5-year weekly optimal look-back period and frequency. For q=0 the rank is based on (shrunk) beta, while for q=1 the rank is based purely on a stock's volatility.



The previous results are based on combining the shrinkage beta estimate with volatility for a varying q parameter. Figure 2 shows the row corresponding to the three-year weekly forward period for varying q when for betas with two-parameter shrinkage and conventionally estimated betas.[12] It shows that for the two-parameter shrunk beta the volatility is below 10.8% per annum for a range of $q = 0$ to $q = 0.6$, with the lowest volatility at $q = 0.1$, as shown in Table 4. For the conventional beta the range is narrower and starts later, with $q = 0.4$ to $q = 0.7$. In practical terms, this means that a half-half combination of a conventional beta and volatility leads to the lowest portfolio volatility, just below 10.8% per annum, indicating that correlation is indeed and important element to create a low-volatility portfolio. However, sorting stocks into portfolios based on their conventional betas gives too much weight to the correlation, which is less accurately forecasted. Reducing the weight of the correlation can be achieved in two different ways that lead to similar results: (1) shrink correlations more to their cross-sectional average than volatilities, (2) combine the conventional not-shrunk beta with a pure volatility estimate. Both approaches lead to similar portfolio volatilities, just below 10.8% in our specific application.

Figure 3: Comparing portfolio volatilities of different methods



[12] To examine whether this combined strategy benefits from the shrunk beta introduced, the Appendix Table A3 includes the D1 portfolio statistics of the combined strategy based on a regular beta estimate that is not shrunk (as in Table 4).

This result is illustrated in Figure 3, in which the two columns on the left are the portfolio volatilities when ranking is based on only conventional beta or only volatility, which are close to 11.0% per annum. The two right columns are the portfolio volatilities when ranking is based on two-parameter shrunk betas or an equally weighted combination of z-scores of not-shrunk betas and volatility, leading to similar outcomes just below 10.8% per annum.

6. Conclusion

We challenge the common approach of simply shrinking security betas to their sample mean of one. We disentangle correlations from relative volatilities and allow different shrinkage parameters to better predict betas. The empirical results confirm that beta prediction errors are significantly lower when correlations are shrunk more than relative volatilities, especially in cases where the betas are difficult to forecast.

In our application of low-risk portfolios, we find that portfolios ranked on betas where correlations have been shrunk more than relative volatilities have lower ex-post betas, albeit not statistically significant. In the debate whether low-beta or low-volatility stocks result in portfolios with the lowest risk, we find that an equally weighted combination performs best. This combination illustrates how our results on two-parameter beta shrinkage should be interpreted: using correlation helps for low-risk portfolio management, but since it is less accurately estimated than volatilities, it has to be handled with care.

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Appendix

Figure A1: Total number of stocks in our sample

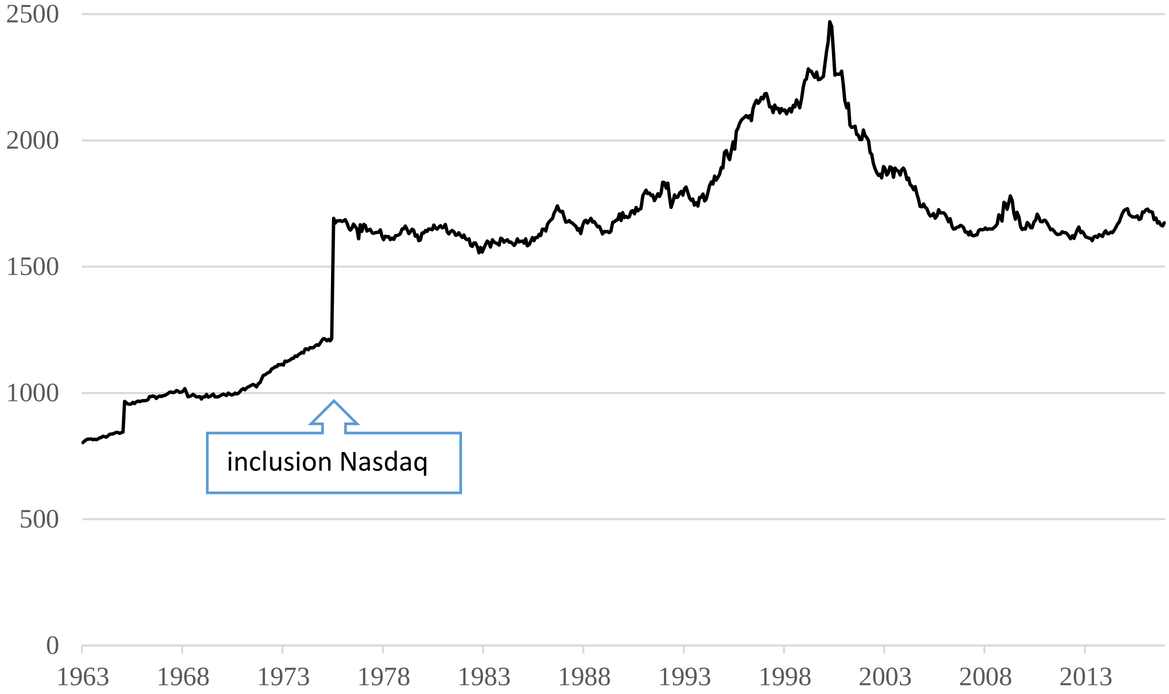

Table A1: Predicting correlation, relative volatility, and beta

Forward periods on which the correlation, relative volatility, and beta are estimated range from 1 month (1M) to 5 years (5Y) and use daily (D), weekly (W), or monthly (M) data frequencies. Numbers in bold with grey background have the lowest mean absolute deviation (MAD) for a forward period and frequency combination per column.

Panel A: MAD Correlation estimates

	Forward	**1M**	**6M**	**1Y**	**3Y**	**5Y**	**1Y**	**3Y**	**5Y**	**1Y**	**3Y**	**5Y**
Look-back	Freq	D	D	D	D	D	W	W	W	M	M	M
1M	D	0.225	0.184	0.181	0.183	0.186	0.214	0.207	0.208	0.299	0.260	0.253
6M	D	0.187	0.123	0.115	0.117	0.119	0.157	0.146	0.147	0.260	0.212	0.203
1Y	D	**0.185**	**0.117**	**0.108**	0.108	0.110	0.151	0.137	0.139	0.256	0.204	0.196
3Y	D	0.188	0.120	0.110	**0.104**	**0.105**	0.148	0.129	0.130	0.252	0.196	0.184
5Y	D	0.190	0.122	0.112	0.106	0.106	0.150	0.131	0.131	0.252	0.195	0.183
1Y	W	0.213	0.155	0.147	0.143	0.143	0.161	0.145	0.142	0.244	0.188	0.176
3Y	W	0.206	0.144	0.135	0.125	0.125	0.145	0.120	0.118	0.232	0.166	0.153
5Y	W	0.205	0.143	0.133	0.124	0.122	**0.143**	**0.118**	**0.115**	**0.229**	**0.164**	**0.150**
1Y	M	0.298	0.259	0.253	0.253	0.253	0.243	0.234	0.233	0.282	0.239	0.231
3Y	M	0.262	0.215	0.208	0.203	0.203	0.193	0.174	0.174	0.239	0.179	0.169
5Y	M	0.256	0.208	0.201	0.195	0.195	0.183	0.163	0.162	0.230	0.167	0.155

Panel B: MAD Relative Volatility Estimates

	Forward	**1M**	**6M**	**1Y**	**3Y**	**5Y**	**1Y**	**3Y**	**5Y**	**1Y**	**3Y**	**5Y**
Look-back	Freq	D	D	D	D	D	W	W	W	M	M	M
1M	D	1.201	1.001	0.979	1.022	1.056	1.132	1.164	1.200	1.352	1.344	1.367
6M	D	1.033	0.718	0.668	0.691	0.718	0.807	0.821	0.860	1.046	1.019	1.045
1Y	D	**1.018**	**0.685**	**0.624**	0.636	0.660	0.743	0.755	0.793	0.994	0.953	0.977
3Y	D	1.048	0.702	0.637	**0.604**	0.620	0.708	0.689	0.724	0.936	0.884	0.899
5Y	D	1.058	0.719	0.651	0.609	0.609	0.713	0.693	0.717	0.933	0.885	0.893
1Y	W	1.140	0.789	0.703	0.648	0.644	0.632	0.594	0.608	0.788	0.699	0.702
3Y	W	1.146	0.784	0.701	0.616	0.606	0.590	0.517	0.531	0.723	0.618	0.615
5Y	W	1.139	0.784	0.700	0.611	**0.584**	**0.583**	**0.511**	**0.509**	0.715	0.616	0.605
1Y	M	1.357	1.023	0.941	0.865	0.846	0.755	0.698	0.697	0.787	0.681	0.670
3Y	M	1.332	0.991	0.907	0.809	0.788	0.682	0.593	0.593	0.686	0.553	0.532
5Y	M	1.317	0.982	0.900	0.797	0.764	0.665	0.574	0.554	**0.671**	**0.524**	**0.489**

Panel C: MAD Beta Estimates

	Forward	**1M**	**6M**	**1Y**	**3Y**	**5Y**	**1Y**	**3Y**	**5Y**	**1Y**	**3Y**	**5Y**
Look-back	Freq	D	D	D	D	D	W	W	W	M	M	M
1M	D	0.745	0.570	0.556	0.561	0.574	0.622	0.600	0.604	0.770	0.658	0.646
6M	D	0.584	0.330	0.304	0.304	0.314	0.386	0.347	0.350	0.572	0.427	0.408
1Y	D	**0.575**	**0.310**	0.280	0.271	0.279	0.362	0.314	0.316	0.551	0.397	0.376
3Y	D	0.577	0.312	**0.276**	**0.254**	0.257	0.348	0.289	0.288	0.534	0.371	0.349
5Y	D	0.579	0.320	0.283	0.255	**0.255**	0.348	0.287	0.284	0.527	0.365	0.342
1Y	W	0.623	0.377	0.349	0.336	0.337	0.398	0.352	0.348	0.566	0.416	0.392
3Y	W	0.597	0.338	0.303	0.277	0.275	0.351	0.289	0.283	0.525	0.358	0.332
5Y	W	0.594	0.338	0.301	0.270	0.265	**0.345**	**0.279**	**0.271**	**0.515**	**0.347**	**0.320**
1Y	M	0.774	0.568	0.542	0.522	0.514	0.565	0.520	0.510	0.689	0.555	0.529
3Y	M	0.657	0.420	0.390	0.363	0.359	0.414	0.355	0.349	0.560	0.398	0.374
5Y	M	0.639	0.399	0.366	0.338	0.333	0.391	0.330	0.322	0.536	0.373	0.348

Table A2: Comparison of two-parameter and one-parameter shrinkage

Look-back period is five-years weekly data. Forward period is three years weekly data. Panel A contains two-parameter shrinkage, with correlation vertically and volatility horizontally. No shrinkage is denoted with 0, where the no shrinkage value of 0.279 is also displayed in Table 1C. Panel B contains one-parameter shrinkage. No shrinkage is denoted with 0, where the no shrinkage values of 0.118 and 0.511 are also displayed in Table 1A and 1B, respectively. Panel C contains the Diebold and Mariano (1995) test for differences in prediction accuracy. In the first hypothesis 0.255 is statistically compared to 0.279, and in the second hypothesis 0.255 is compared to 0.261.

Panel A: MAD of rolling-window beta estimate with two shrinkage factors

c↓ v→	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
0	0.279	0.276	0.276	0.278	0.282	0.288	0.296	0.307	0.319	0.333	0.348
0.1	0.272	0.269	0.269	0.271	0.275	0.282	0.290	0.301	0.313	0.328	0.343
0.2	0.266	0.263	0.263	0.265	0.270	0.276	0.285	0.296	0.309	0.324	0.340
0.3	0.262	0.259	0.259	0.261	0.266	0.273	0.282	0.293	0.306	0.321	0.337
0.4	0.259	0.256	0.256	0.258	0.263	0.270	0.280	0.291	0.304	0.319	0.336
0.5	0.258	0.255	**0.255**	0.258	0.263	0.270	0.279	0.291	0.304	0.319	0.336
0.6	0.258	0.256	0.256	0.258	0.263	0.271	0.280	0.292	0.305	0.320	0.337
0.7	0.260	0.258	0.258	0.261	0.266	0.273	0.283	0.294	0.308	0.323	0.340
0.8	0.264	0.262	0.262	0.265	0.270	0.277	0.287	0.298	0.312	0.327	0.343
0.9	0.269	0.267	0.267	0.270	0.275	0.283	0.292	0.304	0.317	0.332	0.348
1	0.275	0.273	0.274	0.277	0.282	0.289	0.299	0.310	0.323	0.338	0.354

Panel B: MAD with one shrinkage factor

Correlation	0.118	0.115	0.113	0.111	0.111	0.111	0.111	0.113	0.115	0.117	0.121
Relative vol.	0.511	0.501	0.499	0.503	0.515	0.533	0.556	0.585	0.620	0.659	0.702
Beta	0.279	0.269	0.262	**0.261**	0.263	0.270	0.280	0.294	0.311	0.330	0.352

Panel C: Statistical tests	**DM**	**p-val**
No shrinkage versus optimal two-parameter shrinkage	5.96	0.000
Optimal two-parameter versus optimal one-parameter shrinkage	2.41	0.008

Table A3: The impact of including correlations on volatility

Annualized volatility of the D1 portfolio for different forward and frequency combinations (with optimal look-back and frequencies displayed in Table 1). While in Table 5 we had the shrunk beta as the starting point, in this table for q = 0 we have the portfolio ranked on beta without shrinkage, while for q = 1 we have a rank based purely on a stock's volatility. The lowest volatility by row is highlighted with shaded background and bold numbers. *Note that the vertical axis contains the forward-period and forward-frequency, which may lead to the same historical estimation period and frequency, hence resulting in duplicate rows.*

Volatility (%)

Forward	↓Freq → ***q***	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
1M	D	11.18	10.93	10.82	10.72	10.67	**10.59**	10.60	10.62	10.64	10.68	10.74
6M	D	11.18	10.93	10.82	10.72	10.67	**10.59**	10.60	10.62	10.64	10.68	10.74
1Y	D	11.18	10.93	10.82	10.72	10.67	**10.59**	10.60	10.62	10.64	10.68	10.74
3Y	D	11.13	11.00	10.93	10.84	**10.80**	10.84	10.83	10.85	10.90	10.99	11.06
5Y	D	11.32	11.21	11.08	10.98	10.93	10.91	10.85	10.87	10.84	10.91	10.98
1Y	W	11.04	10.91	10.85	10.83	**10.76**	10.79	10.77	10.78	10.80	10.89	10.98
3Y	W	11.04	10.91	10.85	10.83	**10.76**	10.79	10.77	10.78	10.80	10.89	10.98
5Y	W	11.04	10.91	10.85	10.83	**10.76**	10.79	10.77	10.78	10.80	10.89	10.98
1Y	M	11.06	10.91	10.88	**10.82**	10.83	10.85	10.84	10.90	10.87	10.96	11.01
3Y	M	11.06	10.91	10.88	**10.82**	10.83	10.85	10.84	10.90	10.87	10.96	11.01
5Y	M	11.06	10.91	10.88	**10.82**	10.83	10.85	10.84	10.90	10.87	10.96	11.01